Algebraic cycles on Fermat varieties

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A brief review of studies by Prof. Shioda on Fermat varieties

- 1978 : Katsura-Shioda, "On Fermat varieties"
- 1979 : Shioda, "The Hodge conjecture for Fermat varieties"
- 1882 : Shioda, "On the Picard number of a Fermat surface"
- 1983 : A.-Shioda, "Generators of the Neron-Severi group of a Fermat surface"
- 1987 : Shioda, "Some observations on Jacobi sums"
- (lost two decads)
- 2010 : Schütt-Shioda-van Luijk, "Lines on Fermat surfaces"

1. Notation

- Fermat variety/$\mathbb{C}$

$$X_m^n \;:\; x_0^m + x_1^m + \cdots + x_n^m + x_{n+1}^m = 0 \subset \mathbb{P}^{n+1}.$$

- The group

$$G_m^n := \overbrace{\mu_m \times \cdots \times \mu_m}^{n+2} / \text{diagonal}$$

acts on X_m^n in an ovbious manner, where μ_m is the group of m-th roots of unity. We identify the character group $\widehat{G_m^n}$ with the set

$$\{(a_0, \ldots, a_{n+1}) \in (\mathbb{Z}/m\mathbb{Z})^{n+2} \mid a_0 + \cdots + a_{n+2} = 0\}$$

- For any $\alpha \in \widehat{G_m^n}$, let

$$V(\alpha) := \{\xi \in H^n(X_m^n, \mathbb{C}) \mid g^*(\xi) = \alpha(g)\xi \;\; (\forall g \in G_m^n)\}.$$

(α-eigenspace of $H^n(X_m^n, \mathbb{C})$).

2. Hodge cycles on Fermat varieties

Thus

$$H^n(X_m^n, \mathbb{C}) = \bigoplus_{\alpha \in \widehat{G_m^n}} V(\alpha).$$

Let

$$\begin{aligned} \mathfrak{A}_m^n &:= \{(a_0, \ldots, a_{n+1}) \in \widehat{G_m^n} \mid a_i \neq 0 \ (\forall i)\}, \\ 0 &:= (0, \ldots, 0) \in \widehat{G_m^n} \quad \text{(trivial character)}. \end{aligned}$$

Then:

Theorem 1 (Katz, Ogus, Tate, Shioda, ...)**.**
The dimension of $V(\alpha)$ is given by

$$\dim V(\alpha) = \begin{cases} 1 & (\alpha \in \mathfrak{A}_m^n, \textbf{ or } \alpha = 0 \textbf{ if } n \textbf{ is even}), \\ 0 & (\alpha \notin \{0\} \cup \mathfrak{A}_m^n). \end{cases}$$

2. Hodge cycles on Fermat varieties

Suppose n is even, say $n = 2r$.

$$\mathfrak{B}_m^n := \{(a_0, \ldots, a_{n+1} \in \mathfrak{A}_m^n \mid |t \cdot \alpha| = r \ (\forall t \in \mathbb{Z} \text{ with } (t, m) = 1)\},$$

where $|t \cdot \alpha| = \langle ta_0 \rangle_m + \cdots + \langle ta_{n+1} \rangle_m$. (For any $a \in \mathbb{Z}/m\mathbb{Z}$, $\langle a \rangle_m \in \mathbb{Z}$ denotes the unique representative of a such that $0 \leq \langle a \rangle_m < m$.)

Theorem 2. Hodge cycles on X_m^n

$$(H^{r,r}(X_m^n) \cap H^n(X_m^n, \mathbb{Q})) \otimes \mathbb{C} = V(0) \oplus \bigoplus_{\alpha \in \mathfrak{B}_m^n} V(\alpha).$$

Hodge Conjecture : The 1-dimensional space $V(\alpha)$ is generated by algebraic cycles on X_m^n for any $\alpha \in \mathfrak{B}_m^n$.

2. Hodge cycles on Fermat varieties

$n = 2r$: even

Y : a subvariety of X_m^n of codimension r

$[Y]$: the cohomology class of Y

For any $\alpha \in \widehat{G_m^n}$, let

$$\omega_\alpha(Y) = \frac{1}{|G_Y|} \sum_{g \in G_m^n} \overline{\alpha(g)} g^*[Y] \quad \in V(\alpha),$$

where $G_Y = \{g \in G_m^n \mid gY = Y\}$ is the stabilizer of Y in G_m^n.
We say that Y represents α if $\omega_\alpha(Y) \neq 0$.

Problem : Find a subvariety Y representing α for each $\alpha \in \mathfrak{B}_m^n$.

3. Linear spaces on X_m^n

When $\mathfrak{B}_m^n = \mathfrak{D}_m^n$?

Theorem 4 (Parry(?), Shioda 1979). **The following two conditions are equivalent.**
(1) $\mathfrak{B}_m^n = \mathfrak{D}_m^n$ **for all even** n.
(2) m **is a prime or** $m = 4$.

From above theorems we obtain

Corollary 5 (Ran, 1980). **Suppose** m **is a prime or** $m = 4$. **Then** $V(\alpha)$ **is generated by linear spaces on** X_m^n **for any** $\alpha \in \mathfrak{B}_m^n$. **In particular, the Hodge conjecture for** X_m^n **is true for any** n.

3. Linear spaces on X_m^n

When $\mathfrak{B}_m^n = \mathfrak{D}_m^n$ for a fixed n ?

For $n = 2$, we have:

Theorem 6 (Shioda, 1979). **$\mathfrak{B}_m^2 = \mathfrak{D}_m^2$ if and only if one of the following conditions holds:**
(1) $m \leq 4$.
(2) $(m, 6) = 1$.

In general, we have:

Theorem 7 (A-, 1983). **$\mathfrak{B}_m^n = \mathfrak{D}_m^n$ if and only if one of the following conditions holds:**
(1) m **is a prime number or** 4.
(2) $(m, (n+1)!) = 1$.

3. Linear spaces on X_m^n

Corollary 8. **Suppose one of the following conditions holds:**
(1) m **is a prime number or** 4**.**
(2) $(m, (n+1)!) = 1$**.**
Then $V(\alpha)$ **is generated by linear spaces on** X_m^n **for any** $\alpha \in \mathfrak{B}_m^n$**. In particular, the Hodge conjecture for** X_m^n **is true.**

If m is divisible by an odd prime number $l \leq n+1$ and $m \neq l$, then $(m, (n+1)!) > 1$, so $\mathfrak{D}_m^n \subsetneq \mathfrak{B}_m^n$ by Theorem 7.

Therefore, in orer to prove the Hodge conjecture for X_m^n for such m, we need new algebraic cycles.

4. “New” algebraic cycles

l : a prime factor of m such that $d := m/l > 1$,
$a \in \mathbb{Z}/m\mathbb{Z}$ such that $la \neq 0$.

$$\sigma_{l,a} := \begin{cases} (a, a+d, -2a, d) & (l = 2), \\ (a, a+d, a+2d, \ldots, a+(l-1)d, -la) & (l > 2). \end{cases}$$

Proposition 9. **Let $\sigma_{l,a}$ be as above.**

(1) If $l = 2$ and $4a \neq 0$, then $\sigma_{2,a} \in \mathfrak{B}_m^2 - \mathfrak{D}_m^2$.

(2) If $l > 2$, then $\sigma_{l,a} \in \mathfrak{B}_m^{l-1} - \mathfrak{D}_m^{l-1}$.

Remark. If m is divisible by 4, then

$$\sigma_{2,\frac{m}{4}} = \left(\frac{m}{4}, \frac{3m}{4}, \frac{m}{2}, \frac{m}{2}\right) \in \mathfrak{D}_m^2.$$

4. "New" algebraic cycles

- 2-dimensional case:

Theorem 10 (A-, Shioda,1983). **Suppose m is divisible by $l \in \{2,3\}$ and $d := m/l > 1$. Let $\alpha := \sigma_{l,a} \in \mathfrak{B}_m^2$ be the element defined as above. Then:**

(1) If $l = 2$, then the curve C_2 on X_m^2 defined by

$$C_2 : x^d + y^d + \sqrt{-1}z^d = w^2 - \sqrt[d]{2}xy = 0$$

represents α, and $\omega_\alpha(C_2) \cdot \overline{\omega_\alpha(C_2)} = -2m^3$.

(2) If $l = 3$, then the curve C_3 on X_m^2 defined by

$$C_3 : x^d + y^d + z^d = w^3 - \sqrt[d]{-3}xyz = 0$$

represents α, and $\omega_\alpha(C_3) \cdot \overline{\omega_\alpha(C_3)} = -3m^3$.

4. “New” algebraic cycles

The curves defined in Theorem 10 is on the Fermat surface X_m^2. For example, in the case of the curve

$$C_3 : x^d + y^d + z^d = w^3 - \sqrt[d]{-3}xyz = 0,$$

this follows from the formula in high school algebra:

$$x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x + \rho y + \rho^2 z)(x + \rho^2 y + \rho z).$$

Proof. If $m = 3d$, then

$$\begin{aligned} & x^m + y^m + z^m + w^m \\ = \; & (x^d)^3 + (y^d)^3 + (z^d)^3 - 3(x^d y^d z^d) + (w^3)^d + 3(xyz)^d \\ = \; & (x^d + y^d + z^d)(x^d + \rho y^d + \rho^2 z^d)(x^d + \rho^2 y^d + \rho z^d) \\ & \quad + \prod_{\zeta^d = 1} (w^3 - \zeta \sqrt[d]{-3}xyz). \end{aligned}$$

Therefore $C_3 \subset X_m^2$. □

4. "New" algebraic cycles

- heigher dimensional case:

Let l be an odd prime factor of m and suppose that $d := m/l > 1$.

$$\sigma_{l,a} = (a, a+d, a+2d, \ldots, a+(l-1)d, -la) \in \mathfrak{B}_m^n \ (n = l-1).$$

Theorem 11 (A-, 1987). Let l, d, n be as above, and put $n = 2r$. Then the subvariety Y of X_m^n defined by

$$Y : \begin{cases} x_0^{id} + x_1^{id} + \cdots + x_n^{id} = 0 & (1 \leq i \leq r), \\ x_{n+1}^l - \sqrt[d]{-l}\, x_0 x_1 \cdots x_n = 0 \end{cases}$$

represents $\sigma_{l,a}$ for any $a \in \mathbb{Z}/m\mathbb{Z}$ with $la \neq 0$. More precisely, we have $\omega_{\sigma_{l,a}}(Y).\overline{\omega_{\sigma_{l,a}}(Y)} = (-1)^r l^{l-2} m^{n+1}$.

5. Structure of $\mathfrak{B}_m^n$

For any $\alpha = (a_0, \ldots, a_{r+1}) \in \mathfrak{A}_m^r$ and $\beta = (b_0, \ldots, b_{s+1}) \in \mathfrak{A}_m^s$, let

$$\alpha * \beta := (a_0, \ldots, a_{r+1}, b_0, \ldots, b_{s+1})$$

Define sets

$$\mathfrak{A}_m := \bigcup_{\substack{n \geq 0, \\ n:\text{even}}} \mathfrak{A}_m^n, \quad \mathfrak{B}_m := \bigcup_{\substack{n \geq 0, \\ n:\text{even}}} \mathfrak{B}_m^n, \quad \mathfrak{D}_m := \bigcup_{\substack{n \geq 0, \\ n:\text{even}}} \mathfrak{D}_m^n$$

and

$$\mathfrak{S}_m := \left\{ \alpha \in \mathfrak{A}_m \;\middle|\; \begin{array}{l} \alpha * \delta_1 \sim \sigma_{l_1,a_1} * \cdots * \sigma_{l_s,a_s} * \delta_2 \\ (\exists \delta_i \in \mathfrak{D}_m, \exists l_i \mid m, \exists a_i \in \mathbb{Z}/m\mathbb{Z}) \end{array} \right\}$$

Then:

Proposition 12.

(1) $\mathfrak{D}_m \subset \mathfrak{S}_m \subset \mathfrak{B}_m \subset \mathfrak{A}_m$. **(Clear from the definition)**

(2) $\mathfrak{D}_m = \mathfrak{B}_m$ **if and only if** m **is a prime or** 4. **(Theorem 4)**

5. Structure of $\mathfrak{B}_m^n$

Using Shioda's inductive structure, one can prove:

Proposition 13. **For any $\alpha, \beta \in \mathfrak{B}_m$ and $\delta \in \mathfrak{D}_m$, the following assertions hold.**

(1) If both $V(\alpha)$ and $V(\beta)$ are algebraic, then so is $V(\alpha * \beta)$.

(2) If $V(\alpha * \delta)$ is algebraic, so is $V(\alpha)$.

Put

$$\mathfrak{S}_m^n := \mathfrak{S}_m \cap \mathfrak{A}_m^n.$$

From Theorem 11 and Propostion 13 we obtain the following

Corollary 14. **$V(\alpha)$ is algebraic for any $\alpha \in \mathfrak{S}_m$. In particular, if $\mathfrak{B}_m^n = \mathfrak{S}_m^n$, then the Hodge conjecture for X_m^n is true.**

5. Structure of $\mathfrak{B}_m^n$

When $\mathfrak{B}_m = \mathfrak{S}_m$?

Example. The smallest m with $\mathfrak{B}_m \neq \mathfrak{S}_m$ is $m = 12$.
Indeed $\alpha = (1, 9, 8, 6)$ and $\beta = (1, 9, 4, 10)$ are elements of $\mathfrak{B}_{12}^2 - \mathfrak{S}_{12}^2$.

But α and β define the same element "modulo" $\mathfrak{S}_m$ in the following sense: If $\delta = (4, 8, 6, 6) \in \mathfrak{D}_m^2$, then

$$\alpha * \sigma_{2,4} \sim \beta * \delta.$$

Proof.

$$\begin{aligned}
\alpha * \sigma_{2,4} &= (1, 9, 8, 6) * (4, 10, 4, 6) \\
&= (1, 9, 8, 6, 4, 10, 4, 6) \\
&\sim (1, 9, 4, 10, 8, 4, 6, 6) \\
&= (1, 9, 4, 10) * (8, 4, 6, 6) \\
&= \beta * \delta.
\end{aligned}$$

□

5. Structure of $\mathfrak{B}_m^n$

Let

$$g(m) := \begin{cases} \text{the number of prime factors of } m & (m \not\equiv 2 \pmod 4), \\ \text{the number of prime factors of } m/2 & (m \equiv 2 \pmod 4). \end{cases}$$

Theorem 15 (Yamamoto (1973), Kubert(1979?)).
Let $N(m) := 2^{g(m)-1} - 1$. Then:

(1) **There exist $N(m)$ elements $\gamma_1, \ldots, \gamma_{N(m)} \in \mathfrak{B}_m - \mathfrak{S}_m$ with the following properties: For any $\alpha \in \mathfrak{B}_m$, there exist $\sigma \in \mathfrak{S}_m$, $\delta \in \mathfrak{D}_m$ and $i_1, \ldots, i_r \in \{1, 2, \ldots, N(m)\}$ such that**

$$\alpha * \delta \sim \sigma * \gamma_{i_1} * \cdots * \gamma_{i_r}.$$

(2) $\gamma_i * \gamma_i \in \mathfrak{S}_m$ **for any $i = 1, \ldots, N(m)$. In particular, $\alpha * \alpha \in \mathfrak{S}_m$ for any $\alpha \in \mathfrak{B}_m$.**

(3) $\mathfrak{B}_m = \mathfrak{S}_m$ **if and only if $m = l^s$ or $2l^s$ for some prime l.**

5. Structure of $\mathfrak{B}_m^n$

Note that $N(m) > 0 \iff g(m) \geq 2$.

Example. If $m = 12$, then $g(12) = 2$ and $N(12) = 1$.
Let $\gamma = (1, 9, 8, 6)$. Then γ "generates" $\mathfrak{B}_m$ modulo $\mathfrak{S}_m$, and

$$\gamma * \gamma * (2, 10, 5, 7) \sim \sigma_{3,1} * \sigma_{2,1} * \sigma_{2,2}.$$

Indeed,

$$\begin{aligned}
\gamma * \gamma * (2, 10, 5, 7) &= (1, 9, 8, 6) * (1, 9, 8, 6) * (2, 10, 5, 7) \\
&= (1, 9, 8, 6, 1, 9, 8, 6, 2, 10, 5, 7) \\
&= (1, 5, 9, 9, 1, 7, 10, 6, 2, 8, 8, 6) \\
&= (1, 5, 9, 9) * (1, 7, 10, 6) * (2, 8, 8, 6) \\
&= \sigma_{3,1} * \sigma_{2,1} * \sigma_{2,2}.
\end{aligned}$$

□

5. Structure of $\mathfrak{B}_m^n$

Corollary 16. If $m = l^s$ or $2l^s$ for some prime l, then the Hodge conjecture for X_m^n is true for any n.

Corollary 17. If $m = l^s$ or $2l^s$ for some prime l, then the Hodge conjecture for any product of Fermat varieties of degree m is true.

Corollary 18. Let ξ be a Hodge cycle ξ on X_m^n. Then the Hodge cycle $\xi \otimes \xi$ on $X_m^n \times X_m^n$ is algebraic.

6. Lines on Fermat surfaces

$\mathrm{NS}(X_m^2)$: the Néron-Severi group of X_m^2

Recall Theorem 6: $\mathfrak{B}_m^2 = \mathfrak{D}_m^2$ if and only if $m \leq 4$ or $(m, 6) = 1$.

Theorem 19 (Shioda, 1982). **If $m \leq 4$ or $(m, 6) = 1$, then $\mathrm{NS}(X_m^2) \otimes \mathbb{Q}$ is generated by lines on X_m^2.**

Question(Shioda, 1982) : Is $\mathrm{NS}(X_m^2)$ generated by lines on X_m^2 if $(m, 6) = 1$?

Theorem 20 (Schütt-Shioda-van Luijk, 2010). **If $m \leq 100$ and $(m, 6) = 1$, then $\mathrm{NS}(X_m^2)$ is generated by lines on X_m^2.**

6. Lines on Fermat surfaces

Theorem 21 (Degtyarev, 2014). **If $m \leq 4$ or $(m, 6) = 1$, then $\mathrm{NS}(X_m^2)$ is generated by lines on X_m^2.**

- Schütt-Shioda-van Luijk used a supersingular reduction tequnique.
- Degtyarev's proof is topological.
- We can give a new proof using Lim's theorem on the structure of the endomorphism group $\mathrm{End}(J_m)$ of the jacobian variety J_m of the Fermat curve X_m^1.

7. Outline of our proof

Suppose $(m, 6) = 1$ and fix a primitive m-th root of unity ζ.
Let $\mathcal{L}$ be the submodule of $\mathrm{NS}(X_m^2)$ generated by $3m^3$ lines on X_m^2:

$$\begin{aligned} L_{ij}^{(1)} &: \ \zeta^i x + y = \zeta^j z + w = 0, \\ L_{ij}^{(2)} &: \ \zeta^i x + z = \zeta^j y + w = 0, \\ L_{ij}^{(3)} &: \ \zeta^i x + w = \zeta^j y + z = 0. \end{aligned}$$

Proposition 22 (Schütt-Shioda-van Luijk). $3(m-1)(m-2)$ **lines**

$$L_{ij}^{(\nu)} \quad (\nu = 1, 2, 3,\ 1 \le i \le m-2,\ 2 \le j \le m-2)$$

together with the hyperplane section form a $\mathbb{Q}$-basis of $\mathrm{NS}(X_m^2) \otimes \mathbb{Q}$**.**

Shioda's inductive strucure connecting X_m^2 with $X_m^1 \times X_m^1$ gives us a homomorphism $f : \mathrm{NS}(X_m^2) \longrightarrow \mathrm{End}(J_m)$.

7. Outline of our proof

Let $\Phi : \mathbb{Z}[\mathrm{Aut}(X_m^1)] \longrightarrow \mathrm{End}(J_m)$ the natural map,
and define $\sigma, \tau, \iota \in \mathrm{Aut}(X_m^1)$ by $\sigma(x:y:z) = (\zeta x : y : z)$,
$\tau(x:y:z) = (x:\zeta y:z)$ and $\iota(x:y:z) = (y:x:z)$.

Proposition 23. $f(L_{ij}^{(2)}) = \Phi(\sigma^i \tau^j)$, $f(L_{ij}^{(3)}) = \Phi(\iota\sigma^i\tau^j)$ **and** $f(L_{ij}^{(1)}) = 0$.

Theorem 24 (Lim, 1991). $\Phi(\mathbb{Z}[\sigma,\tau,\iota])$ **is a primitive sublattice of** $\mathrm{End}(J_m)$**, namely** $\mathrm{End}(J_m)/\Phi(\mathbb{Z}[\sigma,\tau,\iota])$ **is torsion-free.**

Using Lim's theorem one can prove that the sublattice generated by $2(m-1)(m-2)$ lines $L_{ij}^{(2)}, L_{ij}^{(3)}$ is primitive in $\mathrm{NS}(X_m^2)$.

By symmetry and Proposition 22, $\mathcal{L}$ is primitive in $\mathrm{NS}(X_m^2)$. □

8. Curves on Fermat surfaces

$\rho := \dim \mathrm{NS}(X_m^2) \otimes \mathbb{Q}$, the Picard number of the Fermat surface X_m^2

Theorem 25. **The Picard number ρ of X_m^2 is given by**

$$\begin{aligned}\rho \;=\; & 3(m-1)(m-2) + \delta_m + 1 + \\ & \quad 24(m/3)^* + 48(m/2)^* + 24\varepsilon(m)\end{aligned}$$

where $\delta_m = 1$ if m is even and $\delta_m = 0$ if m is odd, and for a real number x the symbol $(x)^*$ is defined by

$$(x)^* = \begin{cases} x & (x \in \mathbb{Z}), \\ 0 & (x \notin \mathbb{Z}). \end{cases}$$

and $\varepsilon(m) = \sum_{\substack{d|m \\ 1<d\leq 180}} \Delta(d)$ with some $\Delta(d) \in \mathbb{N}$.

8. Curves on Fermat surfaces

Remark.

(1) In the formula of ρ, we have $|\mathfrak{D}^2_m| = 3(m-1)(m-2) + \delta_m$, and the term δ_m corresponds to $(m/2, m/2, m/2, m/2)$.

(2) The terms $24(m/3)^* + 48(m/2)^*$ correspond to the elements of $\mathfrak{B}^2_m$ listed below:

(i) If $2 \mid m$ and $2a \neq 0$, then $\sigma_{2,a} := \left(a,\ a + \frac{m}{2},\ \frac{m}{2},\ -2a\right) \in \mathfrak{B}^2_m$.

(ii) If $3 \mid m$ and $3a \neq 0$, then $\sigma_{3,a} := \left(a,\ a + \frac{m}{3},\ a + \frac{2m}{3},\ -3a\right) \in \mathfrak{B}^2_m$.

(iii) If $2 \mid m$ and $4a \neq 0$, then $\sigma'_{2,a} := \left(a,\ a + \frac{m}{2},\ 2a + \frac{m}{a},\ -4a\right) \in \mathfrak{B}^2_m$.

(3) The terms $\Delta(d)$'s correspond to $\mathfrak{B}^2_m - \mathfrak{S}^2_m$.

8. Curves on Fermat surfaces

Theorem 26 (=Theorem 10).

(1) **Suppose** $2 \mid m$ **and put** $\alpha := \sigma_{2,a} = (a, a+d, -2a+d, d)$ **with** $d = m/2$. **Then the curve**

$$C : x^d + y^d + \sqrt{-1}z^d = w^2 - \sqrt[d]{2}xy = 0,$$

represents α **and** $\omega_\alpha(C) \cdot \overline{\omega_\alpha(C)} = -2m^3$.

(2) **Suppose** $3 \mid m$ **and put** $\alpha := \sigma_{3,a} = (a, a+d, a+2d, -3a)$ **with** $d = m/3$. **Then the curve**

$$C : x^d + y^d + z^d = w^3 - \sqrt[d]{-3}xyz = 0,$$

represents α **and** $\omega_\alpha(C) \cdot \overline{\omega_\alpha(C)} = -3m^3$.

8. Curves on Fermat surfaces

Theorem 27 (A.-Shioda, 1983+2010). **Suppose** $2 \mid m$ **and put** $\alpha := \sigma'_{2,a} = (a, a+d, 2a+d, -4a) \in \mathfrak{B}^2_m$ **with** $d = m/2$.

(1) If $4 \mid m$**, then the curve** C **defined by**

$$C : \begin{cases} x^d + y^d + \sqrt{2}(xy)^{d/2} + \sqrt{-1}z^d = 0, \\ w^4 - \sqrt[d]{-8}xyz^2 = 0 \end{cases}$$

represents α**.**

(2) If $4 \nmid m$**, then the curve** C **defined by**

$$C : \left\{ \begin{array}{l} (x^d + y^d + \sqrt{-1}z^d)z - \frac{\sqrt{2}}{\sqrt[m]{-8}}(xy)^{(d-1)/2}w^2 = 0, \\ w^4 - \sqrt[d]{-8}xyz^2 = 0 \end{array} \right\} \cap X^2_m$$

represents α**.**

More precisely, we have $\omega_\alpha(C) \cdot \overline{\omega_\alpha(C)} = -4m^3$**.**

8. Curves on Fermat surfaces

Theorem 28. **Let** $a = \frac{\sqrt{\sqrt{3}-1}}{\sqrt[8]{12}}$**.**

(1) The curve C **on** X_{12}^2 **defined by**

$$C : \left\{ \begin{array}{l} x^6 + x^2z^4 + a^2y^6 = 0, \\ xw^6 - a\{(1+\sqrt{3})x^4 + z^4\}y^3 = 0 \end{array} \right\} \cap X_m^2$$

repersents $\alpha = (1,9,8,6)$**, and** $\omega_\alpha(C).\overline{\omega_\alpha(C)} = -3 \cdot (12)^3$**.**

(2) The curve D **on** X_{12}^2 **defined by**

$$D : \left\{ \begin{array}{l} x^6 - \sqrt[3]{2}x^2z^2w^2 + a^2z^6 = 0, \\ x(z^6 + w^6) - a\{(1+\sqrt{3})x^4 - \sqrt[3]{2}z^2w^2\}y^3 = 0 \end{array} \right\} \cap X_m^2$$

represents $\beta = (1,9,4,10)$**, and** $\omega_\beta(D).\overline{\omega_\beta(D)} = -24 \cdot (12)^3$**.**

(2) : Jan. 28, 2015, and (1) : Jan. 28, 1984.