Lines in Fermat hypersurface and $\mathcal{M}_{0,n}$

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Contents

Variety of lines in a hypersurface (Short review)

$X \subset \mathbb{P}^{n+1}$: a smooth hypersurface of degree d.
$Gr = Grass(n+1, 1)$: the Grassmann variety of lines in $\mathbb{P}^{n+1}$

Definition

The Fano variety of lines $F(X)$ of X is defined by

$$\{l \in Gr \mid l \subset X\}.$$

Fact (Barth-Van de Ven)

$2n - d - 1 > 0$,
$X \in \mathbb{P}^{n+1}$: *generic of degree* d.
$\Rightarrow$ *the Fano variety* $F(X)$ *is smooth,*
$\dim(F(X)) = 2n - d - 1$.

Variety of lines in a hypersurface (Short review)

Out line of proof

Consider the incidental variety for the universal family.

$V = \mathrm{C}^{n+2}$, S^d_{n+2}= **the space of degree** d **homogeneous polynomial of** $n+2$ **variable.**

$$\mathcal{I} = \{(l, f) \in Gr \times \mathbb{P}(S^d_{n+2}) \mid f\mid_l = 0\} \overset{\pi_1}{\longrightarrow} Gr$$
$$\pi_2 \downarrow$$
$$\mathbb{P}(S^d_{n+2})$$

$\Rightarrow$

1. π_2 **is surjective, and**
2. π_1 **is a projective space bundle of reltaive dimension** $= \dim(S^d_{n+2}) - \dim(S^d_2)$

$\Rightarrow \mathcal{I}$ **is smooth and** $(2n + \dim(S^d_{n+2}) - \dim(S^d_2))$**-dimensional**

$\overset{\text{generic smoothness}}{\Rightarrow}$ **dimension of** $F(X)$=$2n-d-1$ **and smooth.**

Open part $F^0(X)$ of Fano variety

$(X_0 : \cdots : X_{n+1})$: **projective coordinate of** $\mathbb{P}^{n+1}$.
$H_0, \ldots, H_{n+1} \subset \mathbb{P}^{n+1}$: **hyperplanes** $H_i = \{X_i = 0\}$

Definition

1. $F^0(X) = \{l \in F(X) \mid l \cap H_i \ (i = 0, \ldots, n+1)$ are distinct points$\}$
2. $\mathcal{M}_{0,n+2}$: The moduli space of projective line with distinct $n+2$-points $\Rightarrow$ $(n-1)$-dimensional.

Then we have a map

$$\chi : F^0(X) \to \mathcal{M}_{0,n+2} : l \mapsto \{l \cap H_i\}_{i=0,\ldots,n+1}$$

$d = n \Rightarrow \chi$ **is generically finite of degree** n^{n+1}.

Fermat hypersurface and the main Thorem

X : Fermat hypersurface in $\mathbb{P}^{n+1} = \{(X_0 : \cdots : X_{n+1})\}$ defined by

$$X_0^d + \cdots + X_{n+1}^d = 0.$$

μ_d : The groups of d-th root of unities.
$G = (\mu_d)^{n+2}/\Delta$, $\Delta = \{(\zeta, \ldots, \zeta)\}$.
Then an element $(\zeta_0, \ldots, \zeta_{n+1}) \in G$ acts on X by

$$(X_0 : \cdots : X_{n+1}) \mapsto (\zeta_0 X_0 : \cdots : \zeta_{n+1} X_{n+1})$$

$\Rightarrow G$ also acts on $F^0(X)$.

Theorem

If $d = n$, then G acts on $F^0(X)$ freely, and $F^0(X)/G \simeq \mathcal{M}_{0,n+2}$.
If $d < n$, the map $F^0(X) \to \mathcal{M}_{0,n+2}$ is a quotient of self-fiber product of the "universal" abelian covering of exponent d ramified at $n+2$ points.

Prepation for the Main Theorem

K : a filed with $\mathrm{char}(K) = 0$ and $\mu_d \subset K$. (May not be algebraically closed)
C : a curve over K,
$D = \sum_i a_i(p_i)$ a divisor on C, such that D is principal.
$p_\infty \in C(K) - \mathrm{Supp}(D)$.

Definition (trivialized d-Kummer covering)

1. f : a rational function on C such that
   1. $(f) = D$, and
   2. $f(p_\infty) = 1$.

   The d-Kummer covering of C with the branch index D tivialized at p_∞ is a covering of C defined by $y^d = f$.
2. Also defined for a curve over a scheme.

Rigidified moduli space and the universal curve

Definition

1. Define the rigidified moduli space $\widetilde{\mathcal{M}_{0,n+2}}$ by

$$\{(\lambda_0, \dots, \lambda_{n+1}) \in (\mathbb{P}^1)^{n+2} \mid \lambda_i \neq \lambda_j \text{ for } i \neq j\}.$$

2. Define the universal curve

$$\begin{array}{rcc} \widetilde{\mathcal{U}} = \mathbb{P}^1 \times \widetilde{\mathcal{M}_{0,n+2}} & \longrightarrow & \widetilde{\mathcal{M}_{0,n+2}} \\ (x, \lambda_0, \dots, \lambda_{n+1}) & \longmapsto & (\lambda_0, \dots, \lambda_{n+1}) \end{array}$$

$\exists$ natural compatible actions of $PGL(2)$ on $\widetilde{\mathcal{U}}$ and $\widetilde{\mathcal{M}_{0,n+2}}$. The quotient

$$\widetilde{\mathcal{U}}/PGL(2) \to \widetilde{\mathcal{M}_{0,n+2}}/PGL(2) \simeq \mathcal{M}_{0,n+2}$$

is the universal curve over $\mathcal{M}_{0,n+2}$.

Kummer coverings

(1) We fix $\infty \in \mathbb{P}^1$.
The universal section $\widetilde{\mathcal{M}_{0,n+2}} \to \widetilde{\mathcal{U}}$ is also denoted by λ_i.
$Kum_{0,i} \to \widetilde{\mathcal{U}}$: the d-th Kummer covering of $\widetilde{\mathcal{U}}$ for branching $(\lambda_i) - (\lambda_0)$ trivialized by ∞.
$Kum := Kum_{0,1} \times_{\widetilde{\mathcal{U}}} \cdots \times_{\widetilde{\mathcal{U}}} Kum_{0,n+1}$
($Kum_{0,i}$ is defined by $y_i^d = \dfrac{x-\lambda_i}{x-\lambda_0}$, where x is a coordinate with $x(\infty) = \infty$. We set $\lambda_i = x(\lambda_i)$.)

(2) Let $p \neq i, 0$, We set $\Delta_2 := \{(\lambda_0, \lambda_i) \in (\mathbb{P}^1)^2 \mid \lambda_0 \neq \lambda_i\}$
$\pi_{0,i,p} : \mathbb{P}^1 \times \Delta_2 \to \Delta_2 : (\lambda_p, \lambda_0, \lambda_i) \mapsto (\lambda_0, \lambda_i)$.
λ_0, λ_i : two sections of $\pi_{0,i,p}$.
$\widetilde{\Delta}_{0,i}^p$: d-th Kummer covering of $\mathbb{P}^1 \times \Delta_2$ for $(\lambda_i) - (\lambda_0)$ trivialized at ∞.
$\Delta_{0,i}^p$: the pull-back of $\widetilde{\Delta}_{0,i}^p$ to $\widetilde{\mathcal{M}} = \widetilde{\mathcal{M}_{0,n+2}}$ (with the natural μ_d action.)

Kummer coverings (continued)

(3) $\prod_{\widetilde{\mathcal{M}},p\neq 0,i}\Delta^p_{0,i}\to\widetilde{\mathcal{M}}$: μ_d^n-covering
$\widehat{\mathcal{M}}_{0,i}\to\widetilde{\mathcal{M}}$: the covering corresponding to $\mathrm{Ker}(\mu_d^n\to\mu_d)$.
($\widehat{\mathcal{M}}_{0,i}$ is defined by $\delta_i^d=\displaystyle\prod_{p\neq 0,i}\frac{\lambda_i-\lambda_p}{\lambda_0-\lambda_p}$ using coordinate as before.)
$\widehat{\mathcal{M}}:=\widehat{\mathcal{M}}_{0,1}\times_{\widetilde{\mathcal{M}}}\cdots\times_{\widetilde{\mathcal{M}}}\widehat{\mathcal{M}}_{0,n+1}$: G-covering of $\widetilde{M}$.

(4) $\widetilde{\mathcal{F}}=\underbrace{Kum\times_{\widetilde{\mathcal{M}}}\cdots\times_{\widetilde{\mathcal{M}}}Kum}_{k-\text{times}}\times_{\widetilde{\mathcal{M}}}\widehat{\mathcal{M}}$

This is $G^k\times G$-covering of $\underbrace{\widetilde{\mathcal{U}}\times_{\widetilde{\mathcal{M}}}\cdots\times_{\widetilde{\mathcal{M}}}\widetilde{\mathcal{U}}}_{k-\text{times}}\times_{\widetilde{\mathcal{M}}}\widehat{\mathcal{M}}$

$\mathcal{F}$: the covering corresponding to
$\mathrm{Ker}(G^k\times G\overset{\sum_{i=1}^k g_i-g_{k+1}}{\longrightarrow}G)$

Statement of the main theorem

We set $k = n - d$.
Then $\dim(\mathcal{F}) = k + n + 2 = 2n - d + 2$.

Theorem

1. *There is a natural action of $PGL(2)$ on $\mathcal{F}$ extending that on $\mathbb{P}^1$. This action commutes with the action of $\mathfrak{S}_k$.*
2. *The quotine $\mathcal{F}/(PGL(2) \times \mathfrak{S}_k)$ is naturally isomorphic to $F^0(X)$ over $\widetilde{\mathcal{M}_{0,n+2}}/PGL(2) \simeq \mathcal{M}_{0,n+2}$. As a consequence, $F^0(X)$ is smooth.*
3. *In particular, if $d = n$, the map $F^0(X) \to \mathcal{M}_{0,n+2}$ is an etale covering with the group G.*

Corollary

The period of $F^0(X)$ can be written as Selberg integrals.

In the case $d = n = 3$ is studied by Roulleau, M.Yoshida.

Proof of the main theorem (1)

[Proof] Let $l \in F^0(X)$. We fix an affile coordinate t of l. Then the map $\mathbb{P}^1 \to X$ can be written as $t \mapsto (X_0(t) : \cdots : X_{n+1}(t))$, where

$$X_0 = \alpha_0 t + \beta_0, \cdots, X_{n+1} = \alpha_{n+1} t + \beta_{n+1}.$$

We consider the quotient by the $GL(2)$-left action on the frame matrix

$$\begin{pmatrix} \alpha_0 & \cdots & \alpha_{n+1} \\ \beta_0 & \cdots & \beta_{n+1} \end{pmatrix}$$

The equality

$$X_0^d + \cdots + X_{n+1}^d = (\alpha_0 t + \beta_0)^d + \cdots + (\alpha_{n+1} t + \beta_{n+1})^d \\ = 0$$

Proof of the main theorem (2)

$\Rightarrow$ **the following equality for** α_i, β_i

$$\begin{cases} \alpha_0^d + \cdots + \alpha_{n+1}^d = 0 \\ \alpha_0^{d-1}\beta_0 + \cdots + \alpha_{n+1}^{d-1}\beta_{n+1} = 0 \\ \ldots \\ \beta_0^d + \cdots + \beta_{n+1}^d = 0 \end{cases}$$

The intersection of l **with** $X_i = 0$ **is equal to** $t = \lambda_i = -\dfrac{\beta_i}{\alpha_i}$

Proof of the main theorem (3)

Consider the fiber of the map $F^0(X) \to \mathcal{M}_{0,n+2}$ at $(\lambda_0, \cdots, \lambda_{n+1})$. The fiber $F^0(X)_\lambda$ is defined by

$$F^0(X)_\lambda \begin{cases} \alpha_0^d + \cdots + \alpha_{n+1}^d = 0 \\ \lambda_0\alpha_0^d + \cdots + \lambda_{n+1}\alpha_{n+1}^d = 0 \\ \cdots \\ \lambda_0^d\alpha_0^d + \cdots + \lambda_{n+1}^d\alpha_{n+1}^d = 0 \end{cases}$$

as a subvariety of $(\alpha_0 : \cdots : \alpha_{n+1}) \in \mathbb{P}^{n+1}$. This is $\underbrace{(d, \ldots, d)}_{(d+1)-\text{times}}$-complete intersecton of $\mathbb{P}^{n+1}$.

Proof of the main theorem (4)
Complete intersection as a product of curves

Still fix $(\lambda_0, \cdots, \lambda_{n+1})$.
$\mathcal{C}_\lambda$: a G-covering of $\mathbb{P}^1$ defined by

$$y_i^d = \frac{x - \lambda_i}{x - \lambda_0} \quad (i = 1, \ldots, n+1)$$

$\Rightarrow$

$$F^0(X)_\lambda \simeq \prod_{p=1}^{k} \mathcal{C}_\lambda^{(p)} / (N \ltimes \mathfrak{S}_k) \tag{1}$$

over $\overline{\mathbb{C}(\lambda_0, \ldots, \lambda_{n+1})}$, where $k = n - d$, and $\mathcal{C}_\lambda^{(p)}$ $(p = 1, \ldots, k)$ are copies of the curve $\mathcal{C}_\lambda$, and $N = \mathrm{Ker}(G^k \xrightarrow{\sum} G)$

Proof of the main theorem (5)

The isomorphism (1) is given by

$$\frac{\alpha_i}{\alpha_0} = \prod_{p=1}^{k} y_i^{(p)} \left(-\frac{\prod_{j\neq 0}(\lambda_j - \lambda_0)}{\prod_{j\neq i}(\lambda_j - \lambda_i)} \right)^{1/d}$$

where coordinates of $\mathcal{C}_\lambda^{(p)}$ are written as $x^{(p)}, y_i^{(p)}$.

Change coordinate of $t \mapsto t' = \dfrac{at+b}{ct+d}$. Then

$$\lambda_i \mapsto \lambda_i' = \frac{a\lambda_i + b}{c\lambda_i + d}, \quad x^{(p)} \mapsto x^{(p)\prime} = \frac{ax^{(p)} + b}{cx^{(p)} + d}$$

Since $d = n - k$, this action can be extended to $\mathcal{F}$ by

$$\frac{\alpha_i}{\alpha_0} \mapsto \frac{\alpha_i'}{\alpha_0'} \frac{c\lambda_i + d}{c\lambda_0 + d}$$

QED

Singularities

X : n-dimensional Fermat hypersurface of degree d.
Assume that $d = n \geq 4$.

$$F^*(X) := \{l \in F(X) \mid l \cap H_i = \text{ finite set for all } i\} \supset F^0(X)$$

Definition

Let $\mathbf{I}$ be a partition $n + 2 = i_1 + \cdots + i_k$ of $n + 2$.

1. The rank $r(\mathbf{I})$ of $\mathbf{I}$.

$$r(\mathbf{I}) = \begin{cases} 2k & \text{at least two multiple component} \\ 2(k-1) & \text{only one multiple component} \\ 2(k-2) & \text{no multiple component} \end{cases}$$

2. $r(l) :=$ the rank of the partition defined by $\{l \cap H_i\}$.
3. $F_r := \{l \in F^* \mid r(l) \leq r\}$

Singularities

Then we have inclusions

$$F_1 \subset F_2 \subset \cdots \subset F_{2n} = F^*.$$

Theorem

The singular set of $F^*(X)$ is equal to F_n. It is non-empty set if $n \geq 4$.