Motivic relative completion

Tomohide TERASOMA (with R.Hain, M.Matsumoto, G.Perlstein)

The University of Tokyo

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Relative completion of a group

$\mathbf{G}$: a group.
$\rho : \mathbf{G} \longrightarrow \mathbf{GL(V)}$: a representation of $\mathbf{G}$.
$\mathbf{S}$=the Zariski closure of the image $\mathbf{Im}(\rho)$

Assumption:
$\mathbf{S}$ is a reductive group.

Index set $\{(\varphi, \mathcal{G}) = (\varphi : \mathbf{G} \longrightarrow \mathcal{G}(\mathbf{K}), \pi : \mathcal{G} \longrightarrow \mathbf{S})\}$ with

1. $\varphi : \mathbf{G} \longrightarrow \mathcal{G}(\mathbf{K})$ is a homomorphism of groups.
2. $\mathcal{G} \longrightarrow \mathbf{S}$ is a surjective homomorphism of group schemes.
3. the kernel of π is a unipotent algebraic group.
4. the image of φ is Zariski dense.

Definition of the relative completion

Define a map of pairs $(\varphi, \mathcal{G}) \longrightarrow (\varphi', \mathcal{G}')$ by the map $\mathcal{G} \longrightarrow \mathcal{G}'$ such that

$$\begin{array}{ccccc} G & \xrightarrow{\varphi} & \mathcal{G} & \xrightarrow{\pi} & S \\ \| & & \downarrow f & & \| \\ G & \xrightarrow{\varphi'} & \mathcal{G}' & \xrightarrow{\pi} & S \end{array}$$

commutes.

Definition (Relative completion of a group)

$G^S := \varprojlim \mathcal{G}$: the relative completion of G

Example

$\rho =$ the trivial representation,

$\Rightarrow G^S$ is the unipotent completion of G.

Categorical characterization relative completion

Let $\mathbf{K}$ be a field of characterisctic zero.
$(\mathbf{K}-\mathbf{vect})$ Cat. of finite dimensional $\mathbf{K}$-vector spaces.
$(\mathbf{Rep_G})$: Cat. of finite dimensional representations of $\mathbf{G}$.
$\Rightarrow$ Tannakian category.

Definition (Tannakian Category)

Abelian $\mathbf{K}$-linear category $\mathcal{T}$ is a Tannakian category iff

1. $\exists$ binlinear operation $\otimes$, $\exists$ unit object $\mathbf{1}$,
2. $\exists$ inner hom $\boldsymbol{\mathcal{H}om(M, N)}$ (=right adjoint of tensor).
3. $\exists$ fiber functor $\boldsymbol{\omega} : \mathcal{T} \to (\mathbf{K}-\mathbf{vect})$, preserving $\otimes$
4. Some conditions (rigidity, ...).

Tannaka fundamental group

Let $\mathcal{T}$: a Tannakian category.
$\omega : \mathcal{T} \to (\mathbf{K} - \mathbf{vect})$: a fiber functor.

Definition (Tannaka fundamental group)

Let $\pi_1(\mathcal{T}, \omega)$ be an algebraic group defined by

$$\pi_1(\mathcal{T}, \omega) = \{(\mathbf{g_T})_\mathbf{T} \in \prod_{\mathbf{T} \in \mathcal{T}} \mathbf{Aut_K}(\omega(\mathbf{T})) \mid$$

the following diagram commutes for all $\theta : \mathbf{T} \to \mathbf{T}'\}$

$$\begin{array}{ccc} \omega(\mathbf{T}) & \stackrel{\omega(\theta)}{\to} & \omega(\mathbf{T}') \\ \mathbf{g_T} \downarrow & & \downarrow \mathbf{g_{T'}} \\ \omega(\mathbf{T}) & \stackrel{\omega(\theta)}{\to} & \omega(\mathbf{T}') \end{array}$$

Tannaka duality

Theorem (Tannak duality)

1. $\boldsymbol{\pi_1(\mathcal{T}, \omega)}$ *is a pro-algebraic group.*
2. *The natural functor*

$$\begin{array}{ccc} \mathcal{T} & \longrightarrow & \mathbf{Rep}^{\mathbf{alg}}(\pi_1(\mathcal{T}, \omega)) \\ \mathbf{M} & \longmapsto & \omega(\mathbf{M}) \end{array}$$

is an equivalence of tensor category.

Subcategories of $(\mathbf{Rep_G})$

$(\mathbf{Rep_S^{alg}}) \subset (\mathbf{Rep_G})$: the full-subcategory generated by the representation $(\mathbf{V}, \rho)$ stable under $\otimes$ and direct summand.
$(\mathbf{Rep_G^{\rho}}) \subset (\mathbf{Rep_G})$: the full-subcategory of $(\mathbf{V}, \theta)$ such that

1. There exists a filtration

$$\mathbf{0} = \mathbf{F^n} \subset \mathbf{F^{n-1}} \subset \cdots \subset \mathbf{F^0} = \mathbf{V}$$

   in $(\mathbf{Rep_G})$ such that
2. $\mathbf{F^i}/\mathbf{F^{i+1}}$ is an object in $(\mathbf{Rep_S^{alg}})$

Characterization

Proposition

The relative completion $\mathbf{G^S}$ *with respect to the map* $\mathbf{G} \longrightarrow \mathbf{S(K)}$ *is canonically isomorphic to* $\boldsymbol{\pi_1(\mathrm{Rep}^{\rho}_{G}, \omega)}$

Problem

1. *Cohomological construction of* $\boldsymbol{\pi_1(\mathrm{Rep}^{\rho}_{G}, \omega)}$ *?*
   Equivalently construction of the Hopf algebra

$$\boldsymbol{\mathcal{O}(\pi_1(\mathrm{Rep}^{\rho}_{G}, \omega)).}$$

2. *How about geometric case ? (local system, variation of Hodge structure,...)*

$\Rightarrow$ relative bar construction.

Conventions for Hopf algebra and co-tensor product

$\mathbf{G}$ an algebraic group
$\Rightarrow$ the structure ring $\mathcal{O} = \mathcal{O}(\mathbf{G})$ has coproduct and counit
$\Rightarrow$ $\mathcal{O}$ is a Hopf algebra.

Fact

(Cat. of left $\mathbf{G}$-modules) $\underset{equivalent}{\sim}$ *(Cat. of left $\mathcal{O}$-comodules)*

Definition (Cotensor product)

$\mathbf{M}$: right $\mathcal{O}$-comodul, $\mathbf{N}$: left $\mathcal{O}$-comodul.
Cotensor product

$$\mathbf{N} \otimes_{\mathcal{O}} \mathbf{M} := \mathbf{Ker}(\mathbf{N} \otimes_{\mathbf{K}} \mathbf{M} \underset{\boldsymbol{\Delta}_{\mathbf{N}} \otimes \mathbf{1} - \mathbf{1} \otimes \boldsymbol{\Delta}_{\mathbf{M}}}{\longrightarrow} \mathbf{N} \otimes \mathcal{O} \otimes \mathbf{M})$$

$\mathcal{O}$ as $\mathcal{O}$-comodule

$\mathcal{O}$ = left and right $\mathbf{G}$-comodule (via right and left multiplications of $\mathbf{G}$).
$\mathbf{L}\mathcal{O} = \mathcal{O}$ as left $\mathcal{O}$-comodule.
$\mathbf{L}\mathcal{O}$ has right $\mathcal{O}$-coaction.
Trivially, we have $\mathbf{L}\mathcal{O} \otimes_{\mathcal{O}} \mathbf{V} \simeq \mathbf{V}$

Introduction to relative bar construction

Depth one case

$$\mathbf{0 \to V_1 \to E \to V_2 \to 0}$$

where $\mathbf{V_1, V_2} \in \mathbf{Rep_S^{alg}} \Rightarrow$ Classified by $\mathbf{Ext^1_G(V_1, V_2)}$.

Universal case: $\mathcal{O} = \mathcal{O}(\mathbf{S})$

$\mathbf{V_1 = L\mathcal{O}^{(1)} = L\mathcal{O}}$, $\mathbf{V_2 = L\mathcal{O}^{(2)} = L\mathcal{O}}$,

(=not finite dimensional)

The group "$\mathbf{Ext^1_G(L\mathcal{O}^{(2)}, L\mathcal{O}^{(1)})}$" has

right $\mathcal{O}$ action induced by that of $\mathbf{L\mathcal{O}^{(1)}}$

left $\mathcal{O}$ action induced by that of $\mathbf{L\mathcal{O}^{(2)}}$

Introduction to relative bar construction

Using reductivity of **S**, we have

$$\begin{aligned}\mathbf{Ext}^1_{\mathbf{G}}(\mathbf{V}_1,\mathbf{V}_2) &= \mathbf{V}_2^* \otimes_{\mathcal{O}} \mathbf{Ext}^1_{\mathbf{G}}(\mathbf{L}\mathcal{O}^{(2)},\mathbf{L}\mathcal{O}^{(1)}) \otimes_{\mathcal{O}} \mathbf{V}_1 \\ &= \mathbf{Hom}_{\mathbf{S}}(\mathbf{V}_2, \mathbf{Ext}^1_{\mathbf{G}}(\mathbf{L}\mathcal{O}^{(2)},\mathbf{L}\mathcal{O}^{(1)}) \otimes_{\mathcal{O}} \mathbf{V}_1)\end{aligned}$$

Thus we have a map

$$\begin{aligned}\mathbf{V}_1 \oplus \mathbf{V}_2 \longrightarrow & (\mathcal{O}^{(1)} \oplus \mathbf{Ext}^1_{\mathbf{G}}(\mathcal{O}^{(2)},\mathcal{O}^{(1)})) \otimes \mathbf{V}_1 \\ & (\mathcal{O}^{(2)} \otimes \mathbf{V}_2)\end{aligned}$$

Introduction to relative bar construction

Universal extension has right $\mathcal{O}$-coaction.

$$\begin{array}{ccccc} 0 \to \mathbf{L}\mathcal{O}^{(1)} & \to & \mathbf{U} & \to & \mathrm{Ext}^1_{\mathbf{G}}(\mathbf{L}\mathcal{O}^{(2)}, \mathbf{L}\mathcal{O}^{(1)}) \to 0 \\ \downarrow & & \downarrow & & \downarrow \\ 0 \to \mathbf{L}\mathcal{O}^{(1)} \otimes \mathcal{O} & \to & \mathbf{U} \otimes \mathcal{O} & \to & \mathrm{Ext}^1_{\mathbf{G}}(\mathbf{L}\mathcal{O}^{(2)}, \mathbf{L}\mathcal{O}^{(1)}) \otimes \mathcal{O} \to 0 \end{array}$$

Here we used an isomorphism

$$\mathbf{L}\mathcal{O}^{(2)} \otimes_{\mathcal{O}} \mathrm{Ext}^1_{\mathbf{G}}(\mathbf{L}\mathcal{O}^{(2)}, \mathbf{L}\mathcal{O}^{(1)}) \simeq \mathrm{Ext}^1_{\mathbf{G}}(\mathbf{L}\mathcal{O}^{(2)}, \mathbf{L}\mathcal{O}^{(1)})$$

and a map

$$\mathbf{L}\mathcal{O}^{(2)} \otimes_{\mathcal{O}} \mathrm{Ext}^1_{\mathbf{G}}(\mathbf{L}\mathcal{O}^{(2)}, \mathbf{L}\mathcal{O}^{(1)}) \to \mathbf{L}\mathcal{O}^{(2)} \otimes \mathrm{Ext}^1_{\mathbf{G}}(\mathbf{L}\mathcal{O}^{(2)}, \mathbf{L}\mathcal{O}^{(1)})$$

to get the universal extension.

Introduction to relative bar construction

Depth two case

$$\mathbf{0} \to \mathbf{E} \to \mathbf{F} \to \mathbf{V_3} \to \mathbf{0} \tag{1}$$

where $\mathbf{E}$ is of depth one.
The universal case $\mathbf{E} = \mathbf{U}$ and $\mathbf{V_3} = \mathbf{L}\mathcal{O}^{(3)}$.
Long exact sequence.

$$\begin{aligned} \cdots &\to \mathbf{Ext}^1_{\mathbf{G}}(\mathbf{L}\mathcal{O}^{(3)}, \mathbf{L}\mathcal{O}^{(1)}) \to \mathbf{Ext}^1_{\mathbf{G}}(\mathbf{L}\mathcal{O}^{(3)}, \mathbf{U}) \\ &\to \mathbf{Ext}^1_{\mathbf{G}}(\mathbf{L}\mathcal{O}^{(3)}, \mathbf{L}\mathcal{O}^{(2)}) \otimes_{\mathcal{O}} \mathbf{Ext}^1_{\mathbf{G}}(\mathbf{L}\mathcal{O}^{(2)}, \mathbf{L}\mathcal{O}^{(1)}) \to \cdots \end{aligned}$$

Pushing out (1) $\Rightarrow$

$$\mathbf{0} \to \mathbf{U} \otimes_{\mathcal{O}} \mathbf{V_1} \to \mathbf{F}' \to \mathbf{V_3} \to \mathbf{0}$$

$\Rightarrow \exists$ an element $\in \mathbf{Hom_G}(\mathbf{V_3}, \mathbf{Ext}^1(\mathbf{L}\mathcal{O}^{(3)}, \mathbf{U}) \otimes_{\mathcal{O}} \mathbf{V_1})$

Introduction to relative bar construction

Thus we have a map

$$\mathbf{V_1 \oplus V_2 \oplus V_3 \longrightarrow (Ext^1_G(L\mathcal{O}, U) \oplus \mathcal{O}) \otimes (V_1 \oplus V_2 \oplus V_3)}$$

$\mathbf{A = R\Gamma_G(\mathcal{O})}$: left $\mathbf{G}$-resolution, right $\mathbf{S}$-equivariant (eg. standard resolution of the group $\mathbf{G}$ with value in $\mathcal{O}$)
$\mathbf{E = Hom_G(A, A)}$: two sided $\mathcal{O}$-comodule with a DGA strcture.

Then $\mathbf{Ext^1(L\mathcal{O}, U)} \oplus \mathcal{O}$ is the cohomology of

$$\mathbf{E \otimes_{\mathcal{O}} E \longrightarrow E \longrightarrow \mathcal{O}}$$

Introduction to relative bar construction

Relative bar complex for a group
$\mathbf{G}$ and $\mathbf{G} \longrightarrow \mathbf{S(K)}$ as before.
$\mathbf{A} = \mathbf{R\Gamma_G}(\mathcal{O})$, $\mathbf{E} = \mathbf{Hom_G(A, A)}$ as before.

Definition ($\mathbf{Bar}_{\mathcal{O}}(\mathcal{O} \mid \mathbf{E} \mid \mathcal{O})$)

We define the relative bar complex $\mathbf{Bar}_{\mathcal{O}}(\mathcal{O} \mid \mathbf{E} \mid \mathcal{O})$ as follows.

$$\cdots \longrightarrow \mathbf{E} \otimes_{\mathcal{O}} \mathbf{E} \otimes_{\mathcal{O}} \mathbf{E} \longrightarrow \mathbf{E} \otimes_{\mathcal{O}} \mathbf{E} \longrightarrow \mathbf{E} \longrightarrow \mathcal{O} \longrightarrow \mathbf{0}$$

Introduction to relative bar construction

Theorem

1. *Category equivalence*

$$(\mathbf{H^0(Bar_{\mathcal{O}}(\mathcal{O} \mid E \mid \mathcal{O})) - \textit{comodules}}) \simeq (\mathbf{Rep_G^\rho})$$

2. $\mathbf{H^0(Bar_{\mathcal{O}}(\mathcal{O} \mid E \mid \mathcal{O}))}$ *is the Hopf algebra of Tannaka fundamental group* $\boldsymbol{\pi_1}(\mathbf{Rep_G^\rho}, \omega)$.

Relative completions for spaces

$\mathbf{X}$: $\mathbf{C}^\infty$-manifold.
$\mathbf{H}$: local system on $\mathbf{X}$.
$\rho : \pi_1(\mathbf{X}, \mathbf{x}) \longrightarrow \mathbf{Aut}(\mathbf{H}_\mathbf{x})$: the monodromy representation.
$\mathbf{S}_\mathbf{x}$: Zariski closure of $\mathbf{Im}(\rho)$. Assume $\mathbf{S}_\mathbf{x}$ is reductive.

Problem

Compute the relative completion for $\pi_1(\mathbf{X}, \mathbf{x}) \longrightarrow \mathbf{S}_\mathbf{x}$.

$\pi_1(\mathbf{X}, \mathbf{x}, \mathbf{y})$: fundamental groupoid
$\pi_1(\mathbf{X}, \mathbf{x}, \mathbf{y}) \longrightarrow \mathbf{Isom}(\mathbf{E}_\mathbf{x}, \mathbf{E}_\mathbf{y})$: monodromy represnetaion
$\mathbf{S}_{\mathbf{x},\mathbf{y}}$: Zariski closure
$\mathcal{O}_{\mathbf{y},\mathbf{x}} = \mathcal{O}(\mathbf{S}_{\mathbf{x},\mathbf{y}})$: coordinate ring
Local system $\mathcal{O}_{\bullet,\bullet}$ on $\mathbf{X} \times \mathbf{X}$ "fiber at $(\mathbf{x}, \mathbf{y}) \in \mathbf{X} \times \mathbf{X}$"$= \mathcal{O}_{\mathbf{x},\mathbf{y}}$.

Relative completions for spaces

$(\mathbf{V}_{1,x}, \rho_1), (\mathbf{V}_{2,x}, \rho_2)$: An algebraic representaion of $\mathbf{S}_x$.
$\Rightarrow$ a local system $\mathcal{V}_1, \mathcal{V}_2$ on $\mathbf{X}$.
An extension

$$0 \to \mathcal{V}_1 \to \mathcal{E} \to \mathcal{V}_2 \to 0$$

is classified by $\mathbf{Ext}^1_{\mathbf{X}}(\mathcal{V}_2, \mathcal{V}_1)$.

$\mathcal{O}_x = \mathcal{O}(\mathbf{S}_x)$ as left $\mathbf{S}_x$ module $= \mathbf{L}\mathcal{O}_x$.
$\Rightarrow$ a local system $\mathcal{L}\mathcal{O}_x$ on $\mathbf{X}$ with $\mathcal{O}_x$ right action

Relation to the universal extension

$$\begin{aligned}\mathbf{Ext}^1_{\mathbf{X}}(\mathcal{V}_2, \mathcal{V}_2) &= \mathbf{V}^*_{2,x} \otimes_{\mathcal{O}_x} \mathbf{Ext}^1_{\mathbf{X}}(\mathcal{L}\mathcal{O}_x, \mathcal{L}\mathcal{O}_x) \otimes_{\mathcal{O}_x} \mathbf{V}_{1,x} \\ &= \mathbf{Hom}_{\mathbf{S}}(\mathbf{V}_{2,x}, \mathbf{Ext}^1_{\mathbf{X}}(\mathcal{L}\mathcal{O}_x, \mathcal{L}\mathcal{O}_x) \otimes_{\mathcal{O}} \mathbf{V}_{1,x})\end{aligned}$$

Relative completions for spaces

$\mathcal{E} = \mathcal{H}\mathbf{om}(\mathcal{L}\mathcal{O}_\mathsf{x}, \mathcal{L}\mathcal{O}_\mathsf{x})$:
local system of endomorphism algebra on $\mathbf{X}$.

$\mathbf{E} = \mathbf{R\Gamma(X}, \mathcal{E}\mathbf{)}$ complex of higher direct image
$\Rightarrow$ $\mathbf{E}$ is a DGA with left-right $\mathcal{O}_\mathsf{x}$-coaction.
$\Rightarrow$ relative bar constrcution $\mathbf{Bar}_{\mathcal{O}_\mathsf{x}}(\mathcal{O}_\mathsf{x} \mid \mathbf{E} \mid \mathcal{O}_\mathsf{x})$.

Relative completions for spaces

Theorem

1. *Category equivalence*

$$(\mathbf{H^0}(\mathbf{Bar}_{\mathcal{O}_x}(\mathcal{O}_\mathbf{x} \mid \mathbf{E} \mid \mathcal{O}_\mathbf{x})) - \textit{comodules}) \simeq (\pi_1(\mathbf{X}, \mathbf{x})^{\mathbf{S}_x} - \textit{modules})$$

2. $\mathbf{H^0}(\mathbf{Bar}_{\mathcal{O}_x}(\mathcal{O}_\mathbf{x} \mid \mathbf{E} \mid \mathcal{O}_\mathbf{x}))$ *is the Hopf algebra of the relative completion* $\pi_1(\mathbf{X}, \mathbf{x})^{\mathbf{S}_x}$ *of the fundamental group* $\pi_1(\mathbf{X}, \mathbf{x})$.

Remark

The relative bar complex $\mathbf{Bar}_{\mathcal{O}_x}(\mathcal{O}_\mathbf{x} \mid \mathbf{E} \mid \mathcal{O}_\mathbf{x})$ *is quasi-isomrophic to that defined by R.Hain (Ann.Sci.ENS.)*

Relative completion for variation of Hodge structures

$\mathbf{X}$ algebraic variety $/\mathbf{C}$.
$\mathbf{H} = \mathbf{H_Q}$ polarized variation of pure Hodge structures (=VHS).
$\mathbf{S_{x,y}}$ the Zariski clusure of $\mathbf{Im}(\pi(\mathbf{X}, \mathbf{x}, \mathbf{y}) \to \mathbf{Isom}(\mathbf{H_x}, \mathbf{H_y}))$
$\Rightarrow \mathbf{S_{x,y}}$: reductive groupoid
$\Rightarrow \mathcal{O}_{\mathbf{y,x}} = \mathcal{O}(\mathbf{S_{x,y}})$: VHS of algebra on $(\mathbf{x}, \mathbf{y}) \in \mathbf{X} \times \mathbf{X}$.

Notation

We use $\boldsymbol{\xi}, \boldsymbol{\eta}, \cdots$ *for variable of VHS,* $\mathbf{x}, \mathbf{y}$ *for fibers.*
$\mathcal{H}\mathbf{om}$= *sheaf hom.*

Example

$\mathcal{O}_{\boldsymbol{\eta},\mathbf{x}}$ =VHS on $\boldsymbol{\eta} \in \mathbf{X}$ with right $\mathcal{O}_{\mathbf{x}}$ co-action.
$\mathcal{H}\mathbf{om}(\mathcal{O}_{\boldsymbol{\eta},\mathbf{x}}, \mathcal{O}_{\boldsymbol{\eta},\mathbf{y}})$= VHS on $\boldsymbol{\eta} \in \mathbf{X}$
with left-$\mathcal{O}_{\mathbf{x}}$, right-$\mathcal{O}_{\mathbf{y}}$ coactions.

Relative completion for variation of Hodge structures

1.

$$\mathbf{M_2} := \mathcal{O}_{\xi_1,x_1} \otimes_{\mathcal{O}_{x_1}} \mathcal{H}om(\mathcal{O}_{\xi_2,x_1}, \mathcal{O}_{\xi_2,x_2})$$
$$\otimes_{\mathcal{O}_{x_2}} \mathcal{H}om(\mathcal{O}_{\xi_3,x_2}, \mathcal{O}_{\xi_3,x_3}) \otimes_{\mathcal{O}_{x_3}} \mathcal{O}_{x_3,\xi_4}$$
$$\mathbf{M_1} := \mathcal{O}_{\eta_1,x_1} \otimes_{\mathcal{O}_{x_1}} \mathcal{H}om(\mathcal{O}_{\eta_2,x_1}, \mathcal{O}_{\eta_2,x_3}) \otimes_{\mathcal{O}_{x_3}} \mathcal{O}_{x_3,\eta_3}$$

are VHS's on $(\xi_1, \xi_2, \xi_3, \xi_4) \in \mathbf{X}^4$ and $(\eta_1, \eta_2, \eta_3) \in \mathbf{X}^3$

2. Let

$$\mathbf{i}_0 : \mathbf{X}^3 \longrightarrow \mathbf{X}^4 : (\eta_1, \eta_2, \eta_3) \mapsto (\eta_1, \eta_1, \eta_2, \eta_3)$$
$$\mathbf{i}_1 : \mathbf{X}^3 \longrightarrow \mathbf{X}^4 : (\eta_1, \eta_2, \eta_3) \mapsto (\eta_1, \eta_2, \eta_2, \eta_3)$$
$$\mathbf{i}_2 : \mathbf{X}^3 \longrightarrow \mathbf{X}^4 : (\eta_1, \eta_2, \eta_3) \mapsto (\eta_1, \eta_2, \eta_3, \eta_3)$$

Then we have the “multiplication” map of VHS's on $\mathbf{X}^3$.

$$\mathbf{i}_k^* \mathbf{M}_2 \longrightarrow \mathbf{M}_1 \quad \mathbf{k} = 0, 1, 2$$

Cosimplicial scheme and relative bar construction of VHS

3. In this way, we have a cosimplicial scheme over $\mathbf{X^2}$:

$$\mathbf{X}^2 \underset{\mathbf{i_0,i_1}}{\longrightarrow} \mathbf{X}^3 \underset{\mathbf{i_0,i_1,i_2}}{\longrightarrow} \mathbf{X}^4 \longrightarrow \cdots$$

and VHS $\mathbf{M_i}$ on $\mathbf{X^{i+2}}$ and a simplicial homomorphism of VHS:

$$\mathbf{i_j^* M_p \rightarrow M_{p-1}} \quad \text{for } \mathbf{j = 0, \dots, p}$$

Structure map

$$\boldsymbol{\pi_i : X^{i+1} \rightarrow X^2 : (\xi_1, \dots, \xi_{i+2}) \mapsto (\xi_1, \xi_{i+2})}$$

4. Take a "strict functorial" higher direct image $\mathbf{R\pi_{i*}}$ (Check, Godement,...). We have a double complex of sheaves on $\mathbf{X^2}$.

$$\boldsymbol{\mathcal{B}\mathrm{ar} : \cdots \rightarrow R\pi_{2*}M_2 \rightarrow R\pi_{1*}M_1 \rightarrow R\pi_{0*}M_0 \rightarrow 0}$$

Main theorem

Theorem (R.Hain-M.Matsumoto-G.Perlstein-T.)

1. *The fiber of* $\mathcal{H}^0(\mathcal{B}\mathrm{ar})$ *at* (x, x) *is canonically isomorphic to the coordinate ring of the relative completion* $\pi_1(X, x)^{S_x}$.
2. $\mathcal{H}^0(\mathcal{B}\mathrm{ar})$ *is an admissible variation of mixed Hodge structure on* X^2.

Remark

1. *The proof is based on the black box of the theory of Hodge modules. We hope to prove without using the theory of Hodge modules.*
2. *If* $S_x = \{e\}$, *then the reltaive completion is nothing but the unilpotent completion. In this case, the main theorem is proved by R.Hain-S.Zucker.*

Hodge representation

Definition

1. Let $\mathbf{H}$ be a mixed Hodge structure.
   A Hodge representation:= a coaction

   $$\mathbf{H} \longrightarrow \mathcal{H}^0(\mathcal{B}\mathbf{ar})_{x,x} \otimes \mathbf{H}$$

   which is a morphism of mixed Hodge structure.
   $\mathbf{Rep}_{\mathbf{Hodge}}(\mathcal{H}^0(\mathcal{B}\mathbf{ar})_{x,x})$: the category of Hodge representations.
2. $(\mathbf{VMHS}_{\mathbf{X}}^{\mathbf{adm}})$: the category of admissible variations of mixed Hodge structures on $\mathbf{X}$.
   $(\mathbf{VMHS}_{\mathbf{X}}^{\mathbf{adm},\mathbf{H}})$: the relative completion (=as local system !) of $(\mathbf{VMHS}_{\mathbf{X}}^{\mathbf{adm}})$ with respect to $\mathbf{H}$.

Corollary to the main theorem

3. $\mathbf{V}$: Hodge representation
   $\Rightarrow$ admissible vaiation of MHS.

$$\mathbf{\Phi(V) := ker(\mathcal{H}^0(\mathcal{B}ar)_{\xi,x} \otimes V \to \mathcal{H}^0(\mathcal{B}ar)_{\xi,x} \otimes \mathcal{H}^0(\mathcal{B}ar)_{x,x} \otimes V)}$$

4. $\mathcal{V}$ a admissible variation mixed Hodge structure.

$$\psi(\mathcal{V}) = \mathbf{Hom_X}(\mathcal{H}^0(\mathcal{B}\mathbf{ar})_{\xi,x}, \mathcal{V})$$

has $\mathcal{H}^0(\mathcal{B}\mathbf{ar})_{\mathbf{x}}$-comodule structure.
$\psi(\mathcal{V})$ has a mixed Hodge structure $\Leftarrow$ fixed part theorem

Corollary to the main theorem

Corollary

1. *$\mathbf{\Phi}$ is a functor from* $\mathbf{Rep}_{\mathbf{Hodge}}(\mathcal{H}^0(\mathcal{B}\mathbf{ar})_{\mathbf{x},\mathbf{x}})$ *to* $(\mathbf{VMHS}_{\mathbf{X}}^{\mathbf{adm}})$.
2. *The functor $\mathbf{\Phi}$ is an equivalence from* $\mathbf{Rep}_{\mathbf{Hodge}}(\mathcal{H}^0(\mathcal{B}\mathbf{ar})_{\mathbf{x},\mathbf{x}})$ *to* $(\mathbf{VMHS}_{\mathbf{X}}^{\mathbf{adm},\mathbf{H}})$.

Remark

$(\mathbf{VMHS}_{\mathbf{X}}^{\mathbf{adm},\mathbf{H}})$ *contains any constant variation of mixed Hodge structure.*

Fiber sequence theorem

Since

$$\pi_1(\mathsf{X},\mathsf{x})^{\mathsf{S_x}} \rtimes \pi_1(\mathsf{MHS},*) \simeq \pi_1(\mathsf{Rep}_{\mathsf{Hodge}}(\mathcal{H}^0(\mathcal{B}\mathsf{ar})_{\mathsf{x},\mathsf{x}}),*)$$

we have the following theorem.
(*= the forgetful functor.)

Theorem (R.Hain-M.Matsumoto-G.Perlstein-T.)

We have the following exact sequence:

$$\begin{aligned} 1 \to \pi_1(\mathsf{X},\mathsf{x})^{\mathsf{S_x}} &\to \pi_1(\mathsf{VHMS}_{\mathsf{X}}^{\mathsf{adm},\mathsf{H}},\mathsf{x}) \\ &\to \pi_1(\mathsf{MHS},*) \to 1 \end{aligned}$$