MOTIVIC BAR COMODULES ASSCOATED TO POLYLOGARITHMS AND AOMOTO POLYLOGARITHM

1. Aomoto polylog and Bloch-Kriz Hopf algebra

1.1. Aomoto polylog and associate mixed Hodge structure.

Setting: $\mathbf{A}^n = \mathbf{A}^n_{\mathbf{C}}$: the n dimensional affine space. $D = \bigcup_{i=1}^n D_i \subset \mathbf{A}^n$, $E = \bigcup_{i=0}^n E_i \subset \mathbf{A}^n$ be hyperplane arrangement (=unidon of hyperplane). In the following, we assume that $D \cup E$ is a normal crossing divisor.

We set

$$H^n_{dR}(\mathbf{A} - D, E - D/\mathbf{C}) = H^n(\mathbf{A}^n - D, j_!\mathbf{C}),$$
$$H^n_B(\mathbf{A} - D, E - D, \mathbf{Q}) = H^n(\mathbf{A}^n - D, j_!\mathbf{Q})$$

where $j : \mathbf{A} - D - D \subset A - D$ is the natural inclusion. We choose a linear form f_i on $\mathbf{A}^n$ such that $D_i = Z(f_i)$(=the zero locus of f_i). Then

$$\Omega = \bigwedge_{i=1}^n d\log f_i$$

defines a relative cohomology group $H^n_{dR}(\mathbf{A} - D, E - D/\mathbf{C})$

We choose a n-dimensional topological cycle $\sigma \in \mathbf{A}^n$ such that $\sigma \subset (\mathbf{A}^n - D)$ and $\partial\sigma \subset E$. Then σ defines an element in the relative homologoy group

$$H_n(\mathbf{A} - D, E - D, \mathbf{Q})$$

The integral

$$L(D, E) = \int_\sigma \Omega$$

is called a Aomoto polylogarithm. Its motivic nature is studied in [BGSV].

Proposition 1.1. (1)

$$H^i_B(\mathbf{A} - D, E - D, \mathbf{Q}) = 0$$

if $i \neq n$.

(2) *We have the weight filtration $W_\bullet$ on $H^i_B(\mathbf{A} - D, E - D, \mathbf{Q})$ and*

$$Gr^W_{2p}(H^n_B(\mathbf{A} - D, E - D, \mathbf{Q}))) = \mathbf{Q}(-p)^{b_p}$$

$$b_p := \binom{n}{p}\binom{n}{n-p}$$

So, it is equipped with a (natural) mixed Tate Hodge structure.

We set $A_{Hg}(D, E) = H^n_B(\mathbf{A} - D, E - D, \mathbf{Q}))$ (a mixed Hodge structure.)

1.2. **Bloch Kriz Hopf algebra.**

$$Z^i(\mathrm{pt}, j) = \text{Bloch (cubical) } j\text{-th cycle complex of codimension}=i$$
$$+ \text{ (admissiblity condition)}$$

Admissiblity = admissiblity for the faces of $\square^i = (\mathbf{P}^1 - \{\infty\})^n$. We set

$$N^i(j) = Z^j(\mathrm{pt}, 2j-i), \quad N(j) = N^\bullet(j) = \oplus_i N^i(j), \quad N = N^\bullet = \oplus_j N(j)$$

Then N becomes a DGA.

Augmentation:$\epsilon : N \to N^0(0) = \mathbf{Q}$. becomes an augmentation. Bar complex $B(N)$

$$\begin{array}{ccccccccc} B(N) = & \mathbf{Q} & \oplus & N & \oplus & N\otimes N & \oplus & N\otimes N\otimes N & \oplus\cdots \\ & \| & & \| & & \| & & \| & \\ & \mathbf{Q} & & [N] & & [N \mid N] & & [N \mid N \mid N] & \cdots \end{array}$$

+ bar differential Then

Proposition 1.2. *$H_{BK} = H^0(B(N))$ becomes a Hopf algebra, which is called Bloch-Kriz Hopf algebra.*

Definition 1.3. *A comodule over H_{BK} is called a mixed Tate motif (in the sense of Bloch-Kriz). The category of mixed Tate motives are denoted by (MTM). A natrual realization functor to the category $MTHS$ of mixed Tate Hodge strucures*

$$\omega_{Hg} : (MTM) \to (MTHS)$$

is defined in [BK].

Problem 1.4. *It there an object $A_M(D,E)$ in (MTM) such that*

$$\omega_{Hg}(A_M(D,E)) = A_{Hg}(D,E)$$

The related problem for polylogarithms [BK], [T], multiple polylogarithms [GGL], [FJ] are previsouly studied.

1.3. **Main theorem.** We have the following main theorem.

Theorem 1.5. *There exists explicit (combinatorial) construction of $A_M(D,E)$. It is constructed as follows.*

Underlying space :$V = \oplus V_{-p}$ with $\dim V_{-p} = \mathbf{Q}^{b_p}$. *We define ∇*

$$\nabla : V = V_{-p} \to V_{-q} \otimes N^1(p-q).$$

with the coefficients in Bloch cycle algebra which can be computed combinatorially.

A coefficient is obtained by using a linear sequence of codimension one affine inclusions, for example

$$S : W_0 \xrightarrow{f_0} W_1 \xrightarrow{f_1} W_2 \xleftarrow{f_2} W_3 \xrightarrow{f_3} W_4 \xleftarrow{f_4} W_5$$

whose arrows are codimension one affine inclusions.

$$\theta_i =: \square^1 \times W_i \to W_i$$

is a homotopy to a contraction to a point, i.e. by chooseing a coordinate $\theta_i(t_i, x_i) = t_i x_i$. Note that $\theta_i(1, *) = \mathrm{id}$ and $\theta_i(0, *) = 0$.

For this diagram, we associate the equation

$$x_0 = f_0^{-1}\theta_1 f_1^{-1}\theta_2 f_2 \theta_3 f_3^{-1}\theta_4 f_4(0)$$

This is really a set of equations. We rewrite this using $x = f^{-1}(y) \leftrightarrow f(x) = y$.

$$\begin{aligned} f_0(x_0) =&\theta_1 f_1^{-1}\theta_2 f_2 \theta_3 f_3^{-1}\theta_4 f_4(0) \\ =&\theta_1(x_1) \\ f_1(x_1) =&\theta_2 f_2 \theta_3 f_3^{-1}\theta_4 f_4(0) \\ =&\theta_2 f_2 \theta_3(x_3) \\ f_3(x_3) =&\theta_4 f_4(0) \end{aligned}$$

By forgetting the variable x_0, we obtain a codimension 3 subvariety Z_S in $\square^4 = \{(t_1, \cdots, t_4)\}$. Thus we get $Z_S \in Z^3(\mathrm{pt}, 4)$.

2. DG complex and homotopical modification

Recall $\mathcal{C}$: DG-category. i.e.

(1) $Hom^\bullet(A, B) = (Hom^\bullet(A, B), \partial)$=Hom-complex for $A, B \in \mathrm{ob}(\mathcal{C})$
(2) $A, B, C \in \mathrm{ob}$, the data called composite homomorphism

$$\mathrm{tot}(Hom^\bullet(B, C) \otimes Hom^\bullet(A, B)) \to Hom^\bullet(A, C)$$

as a homomorphism of complexes are given.
(3) Associativity, etc. for composite homomorphisms.

Definition 2.1. [BK2]

(1) *Pair $(V^\bullet, d_\bullet)$, where $V^i \in \mathrm{ob}(\mathcal{C}), d_{ij} \in Hom(V^j, V^i)$ for $j < i$ is called a DG-complex if $d^2 = \partial d$ (with a suitable sign convention) holds, where $d = \sum_{i>j} d_{ij}$*
(2) *If all d_{ij}'s are degree 1, it is called concentrated at zero.*

Example 2.2. *Four terms case. Then d_{ij} is given as a homomorphisms*

$$V^0 \xrightarrow{d_{10}} V^1 \xrightarrow{d_{21}} V^2 \xrightarrow{d_{32}} V^3$$

and $d_{20} : V^0 \to V^2$,$d_{31} : V^1 \to V^3$,$d_{30} : V^0 \to V^3$. The condition $d^2 = \partial d$ is equal to

$$d_{21}d_{10} + \partial d_{20} = 0, \quad d_{32}d_{21} + \partial d_{31} = 0, \quad d_{32}d_{20} + d_{31}d_{10} + \partial d_{30} = 0,$$

Example 2.3. *$\mathcal{C}_{MTM}$: DG-category of mixed Tate motives. Objects are finite direct sum of $\mathbf{Q}(i)$. Morphisms are characterized by*

$$Hom^\bullet(\mathbf{Q}, \mathbf{Q}(i)) = N^\bullet(i)$$

Composite = multiplication of N.

Proposition 2.4. *Let $V = \{V^\bullet, \nabla_{\bullet,\bullet}\}$ be a DG-complex concentrated at degree 0. We set $V = \oplus V^\bullet$ and $\nabla = \sum \nabla_{\bullet,\bullet}$. We set*

$$\begin{aligned}\Delta_V &= \mathrm{id} + [\nabla] + [\nabla \mid \nabla] + [\nabla \mid \nabla \mid \nabla] + \cdots \\ &: V \to V \otimes B\end{aligned}$$

induces a coproduct sturcutre on V over $H_{BK} = H^0(B)$.

2.1. **Homotopical modification.** We use the situation in the previous subsection. $\{V^\bullet, \alpha_{\bullet,\bullet}\}$ DG-complex in $\mathcal{C}$. Suppose that the following data are given

(1) For each i, $P^i \in \mathcal{C}$.
(2) For each i two closed degree zero morphisms $P^i \xrightarrow{\pi^*} K^i$ and $K^i \xrightarrow{i^*} P^i$ satisfying $i^*\pi^* = id$.
(3) For each i morphisms $\theta \in Hom^{-1}(K^i, K^i)$ satisfying $\pi^* i^* = id - \partial\theta$.

$P^\bullet$ is called a retraction of $K^\bullet$. Sums of i^* and π^* for $\bullet$, it is also denoted by i^*, π^*.

Theorem 2.5. *We set*

$$\begin{aligned}\beta =& i^*\alpha\pi^* + i^*\alpha\theta\alpha\pi^* + i^*\alpha\theta\alpha\theta\alpha\pi^* + \cdots \\ =& \sum_{n\geq 0} i^*\alpha(\theta\alpha)^n\pi^* \\ =& \sum \beta_{\bullet,\bullet}\end{aligned}$$

Then $\{P^\bullet, \beta_{\bullet,\bullet}\}$ is a DG-complex and it is called a homotopical modification of $K^\bullet$.

Example 2.6. *We consider the following situation.*

$$\begin{array}{ccccc} & & \theta & & \\ K^0 & \xrightarrow{\alpha} & K^1 & \xrightarrow{\alpha} & K^2 \\ \pi^* \uparrow & & i^* \downarrow\uparrow \pi^* & & \downarrow i^* \\ K^0 & \xrightarrow{\beta_{10} = i^*\alpha\pi^*} & K^1 & \xrightarrow{\beta_{21} = i^*\alpha\pi^*} & K^2 \end{array}$$

and set $\beta_{20} : K^0 \xrightarrow{i^*\alpha\theta\alpha\pi^*} K^2$. Then $\{P^\bullet, \beta_{\bullet,\bullet}\}$ becomes a DG-complex.

3. Higher algebraic correspondences and homotopical modification

3.1. **Higher correspondences.** X, Y smooth varieties. Algebric correspondence $Z \in Hom(h(Y), h(X))$.

$$\begin{array}{ccc} Z & \subset & X \times Y \\ \text{proper} \searrow & & \downarrow \\ & & X \end{array}$$

Higher algebric correspondence $Z \in Hom^i(h(Y), h(X))$.

$$\begin{array}{ccc} Z & \subset & \square^i \times X \times Y \\ \text{proper} \searrow & & \downarrow \\ & & \square^i \times X \end{array}$$

+ admissibility condition.

Definition 3.1. *"DG-category" with homomorphism complex=*$Hom^\bullet(h(X)(i)[2i], h(Y))$ *is denoted by* $\mathcal{C}_{MM}$ *and called the "DG-category" of mixed motives.*

Remark 3.2. *Admissibility condition is not preserve by composite.*

Example 3.3.
(1) *If* $f : X \to Y$ *is a morphism then the graph is a correspondence* $\Gamma_f = \{(x, f(x))\}$ *it is denoted by* f^*.
(2) *If* $f : X \to Y$ *is a closed immersion, then the transpose* ${}^t\Gamma_f$ *of the graph* Γ_f *is a correspondence* $f_! = {}^t\Gamma_f \in Hom(h(X)(-d)[-2d], h(Y))$ *is called the Gysin map.*

3.2. Aomoto polylog complex.

We recall the situation:

$$D = \cup_i D_i, E = \cup E_j \subset \mathbf{A}^n$$

For $I \subset \{1, \dots, n\}, J \subset \{0, \dots, n\}$, we set

$$D_I = \cap_{i \in I} D_i, \quad E_I = \cap_{j \in J} E_j$$

We consider the following codimension one embeddings:

$$\begin{array}{ccc} D_I \cap E_J & \subset & D_{I-\{p\}} \cap E_J \\ \bigcup & & \\ D_I \cap E_{J \cup \{q\}} & & \end{array}$$

We set $h(I, J) = h(D_I \cap E_J)(-\#I)[-2\#I]$

$$\begin{array}{ccc} h(I, J) & \to & h(I - \{p\}, J) \\ \downarrow & & \\ h(I, J \cup \{q\}) & & \end{array}$$

Thus we have the following "DG-complex" $\{K^\bullet, \alpha\}$ in $\mathcal{C}_{MM}$.

$$\begin{array}{ccccc} \oplus_{\substack{\#I=n \\ \#J=0}} h(I,J) & \to & \oplus_{\substack{\#I=n-1 \\ \#J=0}} h(I,J) & \to & \oplus_{\substack{\#I=n-2 \\ \#J=0}} h(I,J) \\ & & \downarrow & & \downarrow \\ & & \oplus_{\substack{\#I=n-2 \\ \#J=2}} h(I,J) & \to & \oplus_{\substack{\#I=n-2 \\ \#J=1}} h(I,J) \\ & & & & \downarrow \\ & & & & \oplus_{\substack{\#I=n-2 \\ \#J=2}} h(I,J) \end{array}$$

3.3. Retraction.

We fix an identification:

$$\mathbf{A} = \mathbf{A}^{n-\#I-\#J} \simeq D_I \cap E_J$$

The image of 0 is denoted by $p_{I,J}$. Then $\theta \in Hom^{-1}(h(D_I \cap E_J), h(D_I \cap E_J))$ defined by

$$\square^1 \times \mathbf{A} \times \mathbf{A} \supset \{(t, x, tx)\} = \theta$$

give a retraction of $h(D_I \cap E_J)\underset{\pi^*}{\overset{i^*}{\rightleftarrows}}h(p_{I,J})$. (Since $\theta(0,*) = p_{I,J}, \theta(1,*) = \mathrm{id}$ the face map induces $\mathrm{id} - \pi^* i^*$). We set $P(I,J) = h(p_{I,J})(-\#I)[-2\#I]$.

Theorem 3.4. *Let $(P^\bullet, \beta)$ be the homotopy modification of $\{K^\bullet, \alpha_{\bullet,\bullet}\}$. Then homomorphisms satisfies admissiblity conditions. It defines a DG-complex in $\mathcal{C}_{MTM}$*

3.4. **Conclusion, last step.** Now we consider the homotopical modification which is concentrated in 0.

Proposition 3.5. *By resticting β to the vertical subset, the DG-complex is induced by morphisms of vector spaces and actually a complex*

$$\begin{array}{ccc}
\oplus_{\substack{\#I=n \\ \#J=0}} P(I,J) & \oplus_{\substack{\#I=n-1 \\ \#J=0}} P(I,J) & \oplus_{\substack{\#I=n-2 \\ \#J=0}} P(I,J) \\
 & \downarrow & \downarrow \\
 & \oplus_{\substack{\#I=n-2 \\ \#J=2}} P(I,J) & \oplus_{\substack{\#I=n-2 \\ \#J=1}} P(I,J) \\
 & & \downarrow \\
 & & \oplus_{\substack{\#I=n-2 \\ \#J=2}} P(I,J)
\end{array}$$

More over the homology for the vertical arrow vanishes except for the last step.

We again apply the homotopical modification so that it becomes a DG-complex concetnrated at degree zero.

References

[BGSV] Bielinson, Goncharov, Schechtman, Varchencko.
[BK] Bloch, Kriz
[GGL] Gangl, Goncharov. Levin
[FJ] Furusho, Jafari
[T] Totaro
[BK2] Bondal, Kapranov