BOYARSKI PRINCIPLE FOR $\mathcal{D}$-MODULES AND LOESER'S CONJECTURE.

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1. Introduction

Let p be a prime number, d, f natural numbers such that $(d,p) = 1$ and $p^f \equiv 1 \pmod d$. Put $q = p^f$. We choose $\pi \in \overline{\mathbf{Q}_p}$ such that $\pi^{p-1} = -p$. Then the formal power series $\theta(x) = \exp(\pi(x - x^q))$ converges for units in $K = \mathbf{Q}_p(\mu_{q-1})$. Let $O = O_K$ be the interger ring of K. The reduction map $O \to \mathbf{F}_q \simeq O/\mathcal{P}$, where $\mathcal{P}$ is the maximal ideal in O gives a bijection between $\mu_{q-1} \cup \{0\}$ and $\mathbf{F}_q$ and the element $x \in \mu_{q-1} \cup \{0\}$ corresponding to $\bar{x}$ is called the Teichmuller lifting of $\bar{x}$. We define a map $\psi(\bar{x}) = \theta(x)$, where x is the Teichmuller lifting of $\bar{x}$. Then $\psi(\bar{x})$ is an additive character of $\mathbf{F}_q$. We define a Gaussian sum as

$$g(\frac{j}{d}) = \sum_{x \in \mu_{q-1}} \theta(x) x^{j(q-1)/d}.$$

Gross and Koblitz proved the following formula:

$$g(\frac{j}{d}) = \pi^{d_1 + \cdots + d_f} \prod_{i=0}^{f-1} \Gamma_p(< \frac{p^i j}{d} >)$$
$$\frac{j}{d}(q-1) = d_1 + p d_2 + \cdots + p^{f-1} d_f$$

Here $\Gamma_p(s)$ is the Morita's p-adic gamma function. The left hand side is a character sum over a variety over a finite field. The right hand side is a product of special values of a p-adic analytic function. In the paper [B] of Boyarski, he gives an interpretation of this equality in terms of p-adic differential equation with the aid of trace formula of Dwork. This principle is called Boyarski principle. Along this line, this formula is extended to wider class of differential equation with regular singularity in the book [D]. After this progress of Dwork, Loeser proposed the D-module method with Mellin transform to formulate Boyarski principle. Under a conjectural hypothesis (Conjecture 5.3.3 in [L]), he proved that $F - \Delta$ holomoicity is stable under the push forward by morphisms between tori. In this paper, we prove Conjecture 5.3.3 under a certain genericity assumption. (Main Theorem 4.3). The main theorem asserts the locally analyticity of the proto-γ matrix. Using the

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duality theorem, we give an explicit formula of the proto γ matrix for a D-module generated by $\exp(\pi g(x))$ for a non-degenerate Laurent power series $g(x)$. This computation is independent of the Main Theorem. Note that our class contains D-modules with irregular singularities.

Let us explain the contents of this paper. In §2, we prove propositions for ellimination theory, which is used in the rest of this paper. In §3, we recall the formulation of the principle of Boyarski after Loeser, which is based on the theory of $\mathcal{D}$-module and their Mellin transform. In this section, we recall Amice class which is used to formulate the conjectural isomorphism arising from the Frobenius action. In §4, we reformaulate Loeser's conjectrue using overconvergent power series. Due to this reformulation, this conjecture is reduced to the comparson of topologies (Theorem 4.4), which is proved in §4.2. In §5, we explain the relationship between the isomorphism given in Main Theorem and proto γ-matrix. In the last subsection, we explain the relationship between proto γ-matrices and character sums. The equiality in Theorem 5.8 is nothing but the Boyarski principle in original form.

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2. Ellimination theory

2.1. **Regular sequence.** Let K be a field of characteristic zero and $n \geq 1$ be an interger. A Laurent polynomial $g(x) \in K[x_1^{\pm}, \dots, x_n^{\pm}]$ of variable $x_1, \dots, x_n$ is written as

$$g(x) = \sum_{i=1}^{N} a_i M_i,$$

where a_i are non-zero elements in K, M_i are monomials of $x_1, \dots, x_n$ and $M_i \neq M_j$ if $i \neq j$. By using multi index notation, M_i is expressed as x^{w_i} with $w_i \in \mathbf{Z}^n$. We define the Newton polygon $\Delta = \Delta(g)$ of $g = g(x)$ by the convex hull of $\{w_i\} \cup \{0\}$ in $\mathbf{R}^n$. For a natural number $m \in \mathbf{N}$, we define $m\Delta$ as $m\Delta = \{mx \mid x \in \Delta\}$ and put $C(\Delta) = \cup_{m=0}^{\infty} m\Delta$. We define a subring R_Δ of $K[x_1^{\pm}, \dots, x_n^{\pm}]$ and its increasing filtration R_m ($m \in \mathbf{N}$) by

$$R_\Delta = \oplus_{w \in C(\Delta) \cap \mathbf{Z}^n} K x^w$$
$$R_m = \oplus_{w \in m\Delta \cap \mathbf{Z}^n} K x^w$$

Since $R_m \cdot R_l = R_{m+l}$, the multiplication of R induces a bilinear map

$$Gr_m(R_\Delta) \times Gr_l(R_\Delta) \to Gr_{m+l}(R_\Delta),$$

where $Gr_m(R_\Delta) = R_m/R_{m-1}$ for $m \geq 1$ and $Gr_0(R_\Delta) = 1 \cdot K$. Under this multiplication, $Gr(R_\Delta) = \oplus_{m \in \mathbf{N}} Gr_m(R_\Delta)$ comes to be a graded ring. Set $g_i = x_i \dfrac{\partial}{\partial x_i} g$. Then g_i defines an element in $Gr_1(R_\Delta)$.

Proposition 2.1. *If the Laurent polynomial g is generic in the fixed Newton polygon, $g_1, \dots, g_n$ forms a regular sequence of $Gr(R_\Delta)$.*

Proof. Let $H = \{q\mathbf{Q} \mid \mathbf{Z}^n \cap q\Delta \neq \emptyset\}$ and $R_q = \oplus_{w \in q\Delta \cap \mathbf{Z}^n} K x^w$. We put $Gr_q(R_\Delta) = R_q / \cup_{q' < q} R_{q'}$ for $q \in H$ and $Gr^H(R_\Delta) = \oplus_{q \in H} Gr_q(R_\Delta)$. If

$\{g_1, \ldots, g_n\}$ is a regular sequence in $Gr^H(R_\Delta)$, then it is also a regular sequence in $Gr(R_\Delta)$.

We describe the ring $Gr^H(R_\Delta)$ in a combinatorial way. Let $\mathcal{F}$ be the set of face in Δ which does not contain 0 and $\mathcal{F} = \cup_{i=0}^{n-1} \mathcal{F}_i$ the decomposition where $\mathcal{F}_i$ is the set of i-dimensional faces in $\mathcal{F}$. For an element $F \in \mathcal{F}$, the cone generated by F is denoted by $C(F)$. We define a ring $R_F = \oplus_{w \in C(F)} K x^w$ which is considered as a quotient ring of $Gr^H(R_\Delta)$. For elements $F_1 \subset F_2$, we get a natural surjective map $R_{F_2} \to R_{F_1}$. Using this map, we have the following exact sequence.

$$0 \to Gr^H(F_\Delta) \to \oplus_{F \in \mathcal{F}_{n-1}} R_F \to \oplus_{F \in \mathcal{F}_{n-2}} R_F \to \cdots .$$

We recall properties of Koszur complex. Let R be a ring, M be a free R-module of rank n and $h : M \to R$ be a R-homomorphism. We define $Kos(h)^i = Kos_R(h, M)^i = \wedge^{n-i} M$ and $\delta : Kos(h)^i \to Kos(h)^{i+1}$ by $v_1 \wedge \cdots \wedge v_{n-i} \to \sum_{j=1}^{n-i} (-1)^j v_1 \cdot \cdots \cdot \widehat{v_j} \cdots v_{n-i}$. The complex $(Kos(h)^\bullet, \delta)$ is called the Koszur complex for h. A set $\{g_1, \ldots, g_n\}$ is a regular sequence if and only if $H^i(Kos(h)^\bullet) = 0$ for $i = 0, \ldots, n-1$, where $h : R^{\oplus n} \to R$, $h(e_i) = g_i$.

We prove that $H^i(Kos_{R_\Delta}(h, R_\Delta^n)^\bullet) = 0$ for $i = 0, \ldots, n-1$ for $h : R_\Delta^n \to R_\Delta$, $h(e_i) = g_i = E_i(g)$. The induced map $R_F^n \to R_F$ from h is denoted by h_F. Then we have an exact sequenct of complices:

$$0 \to Kos(h)^\bullet \to \oplus_{F \in \mathcal{F}_{n-1}} Kos(h_F)^\bullet \to \oplus_{F \in \mathcal{F}_{n-2}} Kos(h_F)^\bullet \to \cdots .$$

By using standard argument, we have the following spectral sequence:

$$E_1^{p,q} = \oplus_{F \in \mathcal{F}_{n-1-p}} H^q(Kos(h_F)^\bullet) \Rightarrow H^{p+q}(Kos(h)^\bullet)$$

By this spectral sequence, it is enough to prove that $H^q(Kos(h_F)^\bullet) = 0$ if $q < n-p$ and $F \in \mathcal{F}_{n-p-1}$.

We fix $F \in \mathcal{F}_{n-p-1}$. Let $L(F)$ be the linear hull of $C(F)$ in $\mathbf{R}^{n-p}$ and we choose a base $a_1, \ldots, a_{n-p}$ of $L(F) \cap \mathbf{Z}^n$. We put the new variable of R_F as $y_1 = x^{a_1}, \ldots, y_{n-p} = x^{a_{n-p}}$. Then R_F is identified with a subring of $K[y_1^\pm, \ldots, y_{n-p}^\pm]$. We put $F_j = y_j \dfrac{\partial}{\partial y_j}$. Then $E_i(g) \mid_{R_F} = \sum_{j=1}^{n-p} a_{ji} F_j(g \mid_{R_F})$, where $a_j = (a_{j1}, \ldots, a_{jn})$. Let $\phi : R_F^n \to R_F^{n-p}$ defined by $\phi(e_i) = \sum_{j=1}^{n-p} a_{ji} e_j$ and $\tilde{h}_F : R_F^{n-p} \to R_F$ by $\tilde{h}_F(e_j) = F_j(g \mid_{R_F})$. Then we have $h_F = \tilde{h}_F \circ \phi$.

We consider a filtration $F^1 = Ker(\phi) \subset F^0 = R_F^n$ on R_F^n. This filtration induces a filtraion on $Kos(h_F)^q = \wedge^{n-q}(R_F^n)$ defined by $F^s(Kos(h_F)^q) = \wedge^s(F^1) \wedge \wedge^{n-q-s}(F^0)$. Since $Ker(\phi) \subset Ker(h_F)$, we have $\delta(F^s(Kos(h_F)^q)) \subset F^s(Kos(h_F)^{q+1})$, i.e. the filtration F is a filtration of complex $Kos(h_F)^\bullet$. Therefore the associated graded complex $Gr_F^s(Kos(h_F)^\bullet)$ is isomorphic to the complex $\wedge^s(F^1) \wedge Kos(\tilde{h}_F, R_F^{n-p})^\bullet[-p+s]$. As a consequence, we have the following spectral sequence:

$$\begin{aligned} E_1^{a,b} =& \mathbf{H}^{a+b}(Gr_F^a(Kos(h_F, R_F^n)^\bullet)) \\ =& \wedge^a (F^1) \otimes H^{2a+b-p}(Kos(\tilde{h}_F, R_F^{n-p})^\bullet) \Rightarrow H^{a+b}(Kos(h_F)^\bullet). \end{aligned}$$

Therefore if $E_1^{a,b} \neq 0$, then $a \leq p$ and $2a+b-p \geq n-p$. This inequality implies $a+b \geq n-p$. Therefore $H^q(Kos(h_F)^\bullet)=0$ if $q < n-p$. □

Corollary 2.2. *Assume that $g(x)$ is sufficiently generic in the fixed Newton polygon. Let $i_m : Gr_{m-1}^{\oplus n} \to Gr_m$ be a map defined by $i_m(a_1,\dots,a_n) = \sum_{i=1}^n a_i g_i$. Then*

(1) *The cokernel $Coker(i_m)$ of i_m is zero for sufficiently large m.*
(2)

$$\sum_{i=1}^{\infty} \dim Coker(i_m) = n! \cdot vol(\Delta).$$

Proof. 1. Since the Krull dimension of $Gr_m(R_\Delta)$ is n, $Gr(R_\Delta)/(g_1,\dots,g_n)$ is finite dimensional K vector space. Therefore $Coker(i_m)=0$ for a sufficently large m.

2. Since $g_1,\dots,g_n$ is a regular sequence in $Gr(R_\Delta)$, the Koszur complex of homoegeneous maps $Kos(h,R_\Delta)^\bullet$ is exact. Let $c_m = \dim Coker(i_m)$ and $gr_m = \dim Gr_m(R_\Delta)$. Then by the above exact sequence, we have

$$\begin{aligned} \sum_{i=0}^{\infty} c_i t^i &= \sum_{i=0}^{\infty}\sum_{j=0}^{n}(-1)^j \binom{n}{j} gr_{i-j}t^i \\ &= \sum_{m=1}^{\infty} gr_m(1-t)^n t^m \end{aligned} \tag{2.1}$$

Since gr_m is an integer valued polynomila function on m, of degree $n-1$, it can be written as $gr_m = \sum_{i=0}^{n-1} a_i \dfrac{(m)_i}{i!}$, (2.1) is written as $\sum_{i=0}^{\infty} a_i(1-t)^{n-1-i}$. By putting $t=1$, we have $\sum_{i=0}^{\infty} c_i = a_{n-1}$. Since

$$a_{n-1}/(n-1)! = \lim_{m\to\infty} gr_m/m^{n-1} = n \times vol(\Delta),$$

we have the corollary. □

We put $\kappa = n! \cdot vol(\Delta)$. By Corollary 2.2, we can take a homogeneous base $\{N_1,\dots,N_\kappa\}$ of $Gr(R_\Delta)/(g_1,\dots,g_n)$ consisting of monomials. For an element w in $C(\Delta)$, we define $l=v(w)$ by the minimal natural number l such that $w \in l\Delta$. For simplicity, we write $v(N_j)=v(\nu_j)$ where $N_j = x^{\nu_j}$. Then we have $v(w_1+w_2) \leq v(w_1)+v(w_2)$. We study elimination method using Proposition 2.2. Let $K = \mathbf{Q}(a_1,\dots,a_N)$ be the rational function field of algebraically independent variable $a_1,\dots,a_N$.

Proposition 2.3. *There exists a non-zero integer coefficient polynomial $\delta \in \mathbf{Z}[a_i]_i = \mathbf{Z}[a_1,\dots,a_N]$ such that the following statements hold. Let $S = \mathbf{Z}[a_i,\delta^{-1}]$. The S module*

$$\oplus_{v(w)\leq m} Sx^w$$

is denoted by $R_{m,\delta}$.

(1) *For any element* $w \in C(\Delta) \cap \mathbf{Z}n$ *with* $v(w) = l$,

$$x^w = \sum_{i=1}^{n} \eta_i g_i + \sum_{v(N_j) \leq l} c_j N_j,$$

where $\eta_i \in R_{l-1,\delta}$ *and* $c_j \in S$.

(2) *For any element* $w \in C(\Delta) \cap \mathbf{Z}^n$ *with* $v(w) = l$ *there exists a polynomial* $C_i(G_1, \ldots, G_n) \in S[G_1, \ldots, G_n]$ *of degree* l *in* $G_1, \ldots, G_n$ *such that*

$$x^w = \sum_{i=1}^{\kappa} C_i(g_1, \ldots, g_n) N_i.$$

Proof. $\square$

3. Formalism with $\mathcal{D}$-module

3.1. **$\mathcal{D}$-module generated by $\exp(\pi g)$.** In this section, the base field K is a field of characteristic zero. We use the same notations $g(x)$, $\Delta = \Delta(g)$ and R_Δ as in the last section. Let $E_i = x_i \dfrac{\partial}{\partial x_i}$ $(i = 1, \ldots, n)$ be the Eular operator on the variable x_i and $g_i = E_i(g)$. The operator algebra generated by R_Δ and E_i is denoted by $\mathcal{D}_\Delta = R_\Delta[E_i]$. Let π be an element of K. We consider functions of the form $f(x)\exp(\pi g(x))$, where $\exp(\pi g(x))$ is considered as a "formal symbol". (For example, we do not ask whether $\exp(\pi(x + \dfrac{1}{x})) = (\sum_k \dfrac{\pi^k}{k!} x^k)(\sum_l \dfrac{\pi^l}{l!} x^{-l})$ converges or not!) We define the operation of E_i on $f(x)\exp(\pi g(x))$ by

$$E_i(f(x)\exp(\pi g(x))) = (E_i(f(x)) + \pi g_i f(x))\exp(\pi g(x)).$$

Then the R_Δ module $R_\Delta \exp(\pi g(x)) = \{f(x)\exp(\pi g(x)) \mid f(x) \in R_\Delta\}$ comes to be a $\mathcal{D}_\Delta$-module and it is denoted by $\mathcal{M}_\Delta = \mathcal{M}_{\Delta,g}$. By an elementary calclulation shows that $(E_i - \pi g_i)(E_j - \pi g_j) = (E_j - \pi g_j)(E_i - \pi g_i)$. It is easy to see that $\mathcal{M}_\Delta$ is isomorphic to $\mathcal{D}_\Delta / \mathcal{D}_\Delta (E_i - \pi g_i)_i$ where $\mathcal{D}_\Delta (E_i - \pi g_i)_i$ is the left ideal of $\mathcal{D}_\Delta$ generated by $E_i - \pi g_i$ $(i = 1, \ldots, n)$. Note that $(E_i - \pi g_i)(\exp(\pi g(x))) = 0$ $(i = 1, \ldots, n)$.

3.2. **Mellin transform and duality.** In this section we introduce the Mellin transform of $\mathcal{M}$ after Loeser. We introduce a new variable $s = (s_1, \ldots, s_n)$. Let $\mathcal{D}_\Delta[s]$ be the tensor product of $K[s]$ and $\mathcal{D}_\Delta$ over K. We introduce the structre of multiplication on $\mathcal{D}_\Delta[s]$ so that $K[s]$ is central in it. Let $\mathcal{O}_{-s}$ be the $\mathcal{D}_\Delta[s]$ module defined by

$$\mathcal{D}_\Delta[s] / \mathcal{D}_\Delta[s](E_i + s_i)_{i=1,\ldots,n},$$

where $\mathcal{D}_\Delta[s](E_i + s_i)_{i=1,\ldots,n}$ is the left ideal of $\mathcal{D}_\Delta[s]$ generated by $(E_i + s_i)$ $(i = 1, \ldots, n)$. Then $\mathcal{O}_{-s}$ is expressed formally as $R_\Delta[s]x^{-s}$. Since $\mathcal{M}_\Delta$ is $\mathcal{D}_\Delta$-module, $\mathcal{M}_\Delta \otimes_K K[s]$ is a $\mathcal{D}_\Delta[s]$-module.

Proposition 3.1. (1) *The extension group*

$$Ext^n_{\mathcal{D}_\Delta[s]}(\mathcal{O}_{-s}, \mathcal{M}_\Delta \otimes_K K[s])$$

is a $K[s]$ module and naturally isomorhic to $\mathcal{M}_\Delta$.

(2) *Under the above isorphism the action of E_i is identified with the action of $-s_i$.*

Definition 3.2. *The K-algebra generated by R_Δ and s_i with the relation $s_i x_j = x_j(s_i - \delta_{ij})$ is denoted by $\mathcal{D}^\mu_\Delta$. The module $\mathcal{M}_\Delta$ considered as a $\mathcal{D}^\mu_\Delta$-module is denoted by $\mathcal{M}^\mu_\Delta$ and it is called the Mellin transform of $\mathcal{M}_\Delta$.*

Let $s_0 \in K^n$. The residue field of $K[s]$ associated to s_0 is denoted by $\kappa(s_0)$. The base change $\mathcal{O}_{-s} \otimes_{K[s]} \kappa(s_0)$ is denoted by $\mathcal{O}_{-s_0}$.

Proposition 3.3. *We have*

$$\mathcal{M}^\mu_\Delta \otimes_{K[s]} \kappa(s_0) \simeq Ext^n_{\mathcal{D}_\Delta}(\mathcal{O}_{-s_0}, \mathcal{M}_\Delta)$$

and the dimension of the extension group $Ext^n_{\mathcal{D}_\Delta}(\mathcal{O}_{-s_0}, \mathcal{M}_\Delta)$ is equal to κ.

The following $\mathcal{D}_\Delta[s]$-module $\mathcal{O}^*_{-s}$ is useful to consider the dulaity for cohomologies. Set

$$K[s][x_1^\pm, \dots, x_n^\pm]_f = \{\sum_{w \in \mathbf{Z}^n} a_w x^w (\text{ formal sum }) \mid a_w \in K[s]\},$$

$$R_{\Delta^c,f}[s] = \{\sum_{w \in \mathbf{Z}^n - (-C(\Delta))} a_w x^w (\text{ formal sum }) \mid a_w \in K[s]\}.$$

Then $K[s][x_1^\pm, \dots, x_n^\pm]_f x^{-s}$ and $R_{\Delta^c,f}[s]x^{-s}$ are left $\mathcal{D}_\Delta[s]$-modules. The quotient left $\mathcal{D}_\Delta[s]$ module $K[s][x_1^\pm, \dots, x_n^\pm]_f x^{-s}/R_{\Delta^c,f}[s]x^{-s}$ is denoted by $\mathcal{O}^*_{-s}$. The base change $\mathcal{O}^*_{-s} \otimes_{K[s]} \kappa(s_0)$ is denoted by $\mathcal{O}^*_{-s_0}$.

Proposition 3.4. *The extension groups $Ext^n_{\mathcal{D}_\Delta[s]}(\mathcal{O}_{-s}, \mathcal{O}^*_{-s})$ and $Ext^n_{\mathcal{D}_\Delta}(\mathcal{O}_{-s_0}, \mathcal{O}^*_{-s_0})$ are naturally isomorphic to $K[s]$ and K, respectively.*

3.3. **$K[s]$-structure of $\mathcal{M}$.** In this section, we study the structure of $\mathcal{M}_\Delta$ as a $K[s]$ module. In this section we assume that g is sufficiently generic. Let $\{N_j\}_{j=1,\dots,\kappa}$ be a monomial base of $Gr(R_\Delta)$ and $N_j = x^{\nu_j}$.

Proposition 3.5. *By attaching $F(s)N_j \exp(\pi g(x))$ to $F(-E_1, \dots, -E_n)N_j \exp(\pi g(x))$ ($F(s) \in K[s_i]_i$), we have an isomorphism*

$$\oplus K[s_i]_i N_j \exp(\pi g(x)) \xrightarrow{\alpha} R_\Delta \exp(\pi g(x)) \simeq \mathcal{M}_\Delta.$$

In particular $\mathcal{M}_\Delta$ is a free finite $K[s]$-module of rank κ.

Remark 3.6. *In general $\mathcal{M} = K[x_i^\pm] \otimes_{R_\Delta} \mathcal{M}_\Delta$ is not a $K[s]$-module of finite type. If the convex hull of $\{w_i\}_i$ contains the origin as an interior, then $\mathcal{M}_\Delta \simeq \mathcal{M}$ and they are free of finte rank.*

Proof. First we prove the surjectivity of α. □

Proposition 3.7. *The natural pairings*

$$Ext^n_{\mathcal{D}_\Delta[s]}(\mathcal{O}_{-s}, \mathcal{M}_\Delta \otimes K[s]) \times Hom_{\mathcal{D}_\Delta[s]}(\mathcal{M}_\Delta \otimes K[s], \mathcal{O}^*_{-s}) \\ \to Ext^n_{\mathcal{D}_\Delta[s]}(\mathcal{O}_{-s}, \mathcal{O}^*_{-s}) \simeq K[s]$$

$$Ext^n_{\mathcal{D}_\Delta}(\mathcal{O}_{-s_0}, \mathcal{M}_\Delta) \times Hom_{\mathcal{D}_\Delta}(\mathcal{M}_\Delta, \mathcal{O}^*_{-s_0}) \\ \to Ext^n_{\mathcal{D}_\Delta}(\mathcal{O}_{-s_0}, \mathcal{O}^*_{-s_0}) \simeq K$$

are perfect pairings of free $K[s]$ modules and K vector spaces.

Note that $Hom_{\mathcal{D}_\Delta[s]}(\mathcal{M}_\Delta \otimes K[s], \mathcal{O}^*_{-s})$ is isomorphic to

$$\mathcal{K} = Ker((E_i - \pi g_i)_i \mid \mathcal{O}^*_{-s}) = \cap_i (Ker(E_i - \pi g_i) \mid \mathcal{O}^*_{-s})$$

We define elements ξ^*_j in $\mathcal{K}$ such that $(\xi^*_j, N_k) = \delta_{jk}$. Put $S = \mathbf{Z}[a_i, \delta^{-1}]$. Using Proposition 2.3, x^w $(w \in C(\Delta))$ is written as

$$x^w = \sum_{i=1}^{n} \sum_{v(u) \le v(w)-1} b_i(u,w) x^u g_i + \sum_k c_k(w) N_k, \tag{3.1}$$

where $b_i(u,w), c_k(w) \in S$. We choose $b_i(u, \nu_j) = 0$, $c_k(\nu_j) = \delta_{jk}$. For $w \in C(\Delta), v(w) = l$, we define $G_j(w) = G_j(a,s,w) \in \dfrac{1}{\pi^l} S[s]$ inductively by the relation

$$G_j(w) = -\sum_{i=1}^{n} \sum_{v(u) \le l-1} \frac{(s_i + u_i) b_i(u,w)}{\pi} G_j(u) + c_j(w)$$

$$G_j(0) = \begin{cases} 0 & (N_j \neq 1) \\ 1 & (N_j = 1) \end{cases}$$

We define $\xi^*_j = \xi^*_j(a,s,x) = \sum_{w \in C(\Delta)} G_j(w) x^{-w} \cdot x^{-s}$.

Lemma 3.8. (1) $(\xi^*_j, N_k) = \delta_{jk}$.
(2) *The element $\xi^*_j \in \mathcal{O}^*_{-s}$ is contained in $\mathcal{K}$.*

Proof. The relation (1) is a consequence of the choice of $a_j(u,w)$ and $c_j(w)$. To prove (2), it is enough to prove that $(\xi^*_j, \eta) = 0$ for all $\eta \in (E_i - \pi g_i - s_i)(R_\Delta[s])$. Let V be a $K[s]$-submodule of $Im = \sum_i (E_i - \pi g_i - s_i)(R_\Delta[s])$ generated by $\{\sum_{i=1}^n \sum_{v(u) \le v(w)-1} (E_i - \pi g_i - s_i) a_i(u,w) x^u\}$. Then by the definition of ξ^*_j, $(\xi^*_j, \eta) = 0$ for $\eta \in V$. We show that $Im = V$. Let $Im_d = Im \cap R_d$ and $V_d = V \cap R_d$. Since $K[s] \otimes R_d = Im_d \oplus \oplus_{v(N_j) \le d} K[s] N_j$, the equality

$$K[s] \otimes R_d = V_d \oplus \oplus_{v(N_j) \le d} K[s] N_j \tag{3.2}$$

implies $Im_d = V_d$. We show the equality (3.2) by the induction on d. For $w \in C(\Delta), v(w) = d$, we have

$$
\begin{aligned}
x^w &= -\sum_{i=1}^{n} \sum_{v(u)\le d-1} (E_i - \pi g_i - s_i) b_i(u,w) x^u + \sum_{i=1}^{n} \sum_{v(u)\le d-1} (E_i - s_i) b_i(u,w) x^u \\
&\quad + \sum_{v(N_k)\le d} c_k(w) N_k \\
&\quad \in V_d \oplus R_{d-1} \otimes K[s] \oplus \oplus_j N_j K[s]
\end{aligned}
$$

By the above equality and the induction hypothesis, we have $x^w \in V_d \oplus \oplus_{v(N_j)\le d} K[s] N_j$ and this imples the equality (3.2). □

Proof of Proposition 3.7. By Lemma 3.8, $\{\xi_j^*\}_j$ is a dual base of $\{N_j\}_j$. To prove the perfectness, it is enough to prove that $\{\xi_j^*\}_j$ generates $Ker((E_i - \pi g_i - s_i)_i \mid \mathcal{O}^*)$. Let ξ^* be an element in $Ker((E_i - \pi g_i - s_i)_i \mid \mathcal{O}^*)$ and put $b_j = (\xi^*, N_j)$. Then by the equality (3.1), $(\xi^*, x^w) = \sum_{j=1}^{\kappa} b_j (\xi_j^*, x^w)$. Therefore $\xi = \sum_{j=1}^{\kappa} b_j \xi_j^*$. □

3.4. **Completion by Amice class.** In this section we introduce the class of functions $\mathcal{A}(s)$ on $s = (s_1, \dots, s_n)$ after Amice. From this subsection, we assume that $K = \mathbf{Q}_p(\mu_{q-1})$ and $a_i \in K$. The integer ring of K is denoted by O_K. For an element $i = (i_1, \dots, i_n) \in \mathbf{N}^n$, we put $\mid i \mid = \sum_{j=1}^{n} i_j$, $i! = \prod_{j=1}^{n} (i_j)!$ and $(s)_i = \prod_{j=1}^{n} (s_j)_{i_j}$. Let $f(s)$ be a continuous funcition on $\mathbf{Z}_p^n = \{s = (s_1, \dots, s_n) \mid s_i \in \mathbf{Z}_p\}$. By Mahler's theorem, $f(s)$ can be written as

$$
f(s) = \sum_{i\in \mathbf{N}^n} \alpha_i \frac{(s)_i}{i!},
$$

where $\mid \alpha_i \mid_p \to 0$ for $\mid i \mid \to \infty$ and this expression for $f(s)$ is unique. For a continuous function $f(s)$ on $\mathbf{Z}_p^n$ and $s_0 \in \mathbf{Z}_p^n$, $f(s)$ is holomorphic at s_0 if the sequence of polynomilas $\displaystyle\sum_{i\in \mathbf{N}^n, |i|<M} \alpha_i \frac{(s)_i}{i!}$ converges to a holomorphic function at s_0 with a positive convergent radius for $M \to \infty$. A continuous fuction on $\mathbf{Z}_p^n$ is called a locally holomorphic function if it is holomorphic at all $s_0 \in \mathbf{Z}_p^n$.

Proposition 3.9 (Amice)**.** *Let $f(s)$ be a continousou function on $s \in \mathbf{Z}_p^n$. The function $f(s) = \displaystyle\sum_{i\in \mathbf{N}^n} \alpha_i \frac{(s)_i}{i!}$ is locally holomorphic on $\mathbf{Z}_p^n$ if and only if there exist a constant A and B with $0 < B < 1$ such that $\mid \alpha_i \mid_p < A \cdot B^{|i|}$.*

Example 3.10. *For an element w in $\mathbf{Z}_p^n$, we define a function $\chi_{p\mathbf{Z}_p + w}(t)$ by*

$$
\chi_{p\mathbf{Z}_p+w}(t) = \begin{cases} 1 & (t \in p\mathbf{Z}_p + w) \\ 0 & (t \notin p\mathbf{Z}_p + w). \end{cases}
$$

Then $\chi_{p\mathbf{Z}_p-w}(s)=\sum_k \alpha_k \frac{(s+w)_k}{k!}$, *where* $\sum \alpha_k u^k = \frac{u^p}{u^p-(u-1)^p}$. *Therefore* $\chi_{p\mathbf{Z}_p-w}(s)\in\mathcal{A}(s)$.

The K-vector space of continuous functions locally holomorphic on $\mathbf{Z}_p^n$ is denoted by $\mathcal{A}(s)$. The space $\mathcal{A}(s)$ is closed under the multiplication and forms a $K[s]$ algebra. Since $f(s)x^w = x^w f(s-w)$, $\mathcal{A}(s)\otimes_{K[s]}\mathcal{D}^{\mu}_{\Delta}=\mathcal{D}^{\mu}_{\Delta}\otimes_{K[s]}\mathcal{A}(s)$ and it is denoted by $\widehat{\mathcal{D}_\Delta}$. Then the module $\widehat{\mathcal{M}_\Delta}=\mathcal{A}(s)\otimes_{K[s]}\mathcal{M}^{\mu}_{\Delta}$ is a $\widehat{\mathcal{D}_\Delta}$ module in natural way.

3.5. **Pull back by the Frobenius action.** In this section we define the pull back of $\mathcal{M}_\Delta$ by the Frobenius map F. For the general definition of pull back for monomial map, see [L]. We define $R_\Delta^{1/p}$ by

$$\oplus_{w\in C(\Delta)\cup(\frac{1}{p}\mathbf{Z})^n} K y^w,$$

Then R_Δ is identified with a subring of $R_\Delta^{1/p}$ via the rule $y^{pw}=x^w$. The ring of differential operators $\mathcal{D}_\Delta^{1/p}$ is defined as the ring generated by $R_\Delta^{1/p}$ and $F_i = y_i\frac{\partial}{\partial y_i}$. By substituting $F_i=-t_i$, we get a ring $\mathcal{D}_\Delta^{\mu,1/p}$ and put $\widehat{\mathcal{D}_\Delta}^{1/p}=\mathcal{A}(t)\otimes_{K[t]}\mathcal{D}_\Delta^{\mu,1/p}$. The natural inclusion $R_\Delta\subset R_\Delta^{1/p}$ is denoted by F. Then the the morphism of torus associated to F is a monomial map and the $\mathcal{D}_\Delta^{1/p}$ module $F^!\mathcal{M}_\Delta$ and the $\widehat{\mathcal{D}_\Delta}^{1/p}$ module $F^!\widehat{\mathcal{M}_\Delta}$ is defined as in [L]. We recall the defintion of them.As $R_\Delta^{1/p}$ modules, we define

$$F^!\mathcal{M}_\Delta = R_\Delta^{1/p}\otimes_{R_\Delta}\mathcal{M}_\Delta$$
$$F^!\widehat{\mathcal{M}_\Delta} = R_\Delta^{1/p}\otimes_{R_\Delta}\widehat{\mathcal{M}_\Delta}.$$

On these modules, $t_i=-y_i\frac{\partial}{\partial y_i}\in K[t_i]$ and $f(t)\in\mathcal{A}(t)$ acts on $y^w\otimes m$ by $t_i(y^w\otimes m)=y^w\otimes(ps_i-w_i)m$ and $f(t)(y^w\otimes m)=y^w\otimes f(ps-w)m$, respectively. By an easy calculation, we can can show that

$$F^!\mathcal{M}_\Delta=\mathcal{D}_\Delta^{1/p}/\mathcal{D}_\Delta^{1/p}(F_i-p\pi g_i(y^p))=R_\Delta^{1/p}\exp(\pi g(y^p)).$$

4. Main Theorem

4.1. **Loeser's conjecture.** We use the same notation g, $\Delta=\Delta(g)$ etc. The subspace $R_\Delta^\dagger$ of the space of formal power series

$$R_\Delta^\dagger=\{\sum_{w\in C(\Delta)} c_w x^w \mid \text{There exist a positive rational number}$$
$$\epsilon \text{ and a constant } A \text{ such that } |c_w| < Ap^{-\epsilon v(w)}\}$$

is called the space of overconvergent power series with respect to the Newton polygon Δ. It is easy to see that the multiplication on $R_\Delta^\dagger$ is well defined.

For a Laurent polynomial $g(x)=\sum_{i=1}^N a_i x^{w_i}$, we define a Laurent polynomial $g^{(p)}(x)=\sum_{i=1}^N a_i^p x^{w_i}$. Note that $\delta(a_i)\in O_K$, then we have $\delta(a_i^p)\in O_K$. Let $\theta(x)=\exp(\pi(x-x^p))=\sum_{i=0}^\infty c_i x^i$ be the Dwork exponential. Since $ord_p c_i \geq \dfrac{i(p-1)}{p^2}$ (see [K]), the formal power series

$$\exp(\pi(g(y)-g^{(p)}(y^p)))=\prod_{i=1}\theta(a_i y^{w_i})=\sum_{i\in C(\Delta)} c_i(g)y^i$$

is a overconvergent power series. In fact, we have an estimate $ord_p(c_i(g))\geq \dfrac{p-1}{p^2}v(i)$. Let $\mathcal{M}_\Delta^{(p)}$ be the $\mathcal{D}_\Delta$ module generated by $\exp(\pi g^{(p)}(x))$, i.e.

$$\mathcal{M}_\Delta^{(p)}=\mathcal{D}_\Delta/\mathcal{D}_\Delta(E_i-\pi g_i^{(p)}),$$

where $g_i^{(p)}=E_i(g^{(p)})$. The pull back $F^!\mathcal{M}_\Delta^{(p)}$ of $\mathcal{M}_\Delta^{(p)}$ by the Frobenius map is identified with $R_\Delta^{1/p}\exp(\pi(y^p))$, where $R_\Delta^{1/p}=\oplus_{w\in C(\Delta)\cap\frac{1}{p}\mathbf{Z}^n}y^w$. Since R_Δ and $R_\Delta^{1/p}$ are contained in the overconvergent powerseries ring $R_\Delta^\dagger$ and $R_\Delta^{1/p,\dagger}$, respectively, we have two inclusions: $\mathcal{M}_\Delta\to R_\Delta^\dagger\exp(\pi(g(x)))$ and $F^!\mathcal{M}_\Delta^{(p)}\to R_\Delta^{1/p,\dagger}\exp(\pi(g^{(p)}(y^p)))$. It is obvious that the morphism Φ from $R_\Delta^\dagger\exp(\pi(g(x)))$ to $R_\Delta^{1/p,\dagger}\exp(\pi(g^{(p)}(y^p)))$ given by $\Phi(f(x)\exp(\pi g(x)))=[f(y)\exp(\pi(g(y)-g^{(p)}(y^p)))]\exp(\pi g^{(p)}(y^p))$ is an isomorphism of $\mathcal{D}_\Delta$-module. We have the following commutative diagram.

$$\begin{array}{ccccc}
\widehat{\mathcal{M}_\Delta} & \longleftarrow & \mathcal{M}_\Delta=R_\Delta\exp(\pi(g(x))) & \longrightarrow & R_\Delta^\dagger\exp(\pi(g(x))) \\
 & & & & \downarrow\Phi \\
F^!\widehat{\mathcal{M}_\Delta}^{(p)} & \longleftarrow & F^!\mathcal{M}_\Delta^{(p)}=R_\Delta^{1/p}\exp(\pi(g^{(p)}(y^p))) & \longrightarrow & R_\Delta^{1/p,\dagger}\exp(\pi(g^{(p)}(y^p)))
\end{array}$$

We reformulate the Loeser's conjecture as follows.

Conjecture 4.1 (Loeser,[L] 5.3.3)**.** *The morphism Φ induces an isomorphism from $\widehat{\mathcal{M}_\Delta}$ to $F^!\widehat{\mathcal{M}_\Delta}^{(p)}$ for almost all prime numbers.*

Remark 4.2. *Presicely speaking, in [L], Loeser conjectured that there exists Φ without the assumption for the non-deneracy condition. We do not know whether Φ induces an isomorphism for almost all p if $\delta(a_i)=0$.*

The conjecture of Loeser is reduced to the following main theorem.

Theorem 4.3 (Main Theorem)**.** *Suppose that $a_i\in\overline{\mathbf{Q}}$ and $\delta(a_i)\neq 0$. Then for almost all p, the following statements hold.*

(1) *The isomorphism*

$$\alpha:\oplus_{i=1}^\kappa K[s]N_j\exp(\pi g(x))\to R_\Delta\exp(\pi g(x))$$

is extended to the isomorphism

$$\widehat{\alpha} : \oplus_{i=1}^{\kappa} \mathcal{A}(s) N_j \exp(\pi g(x)) \xrightarrow{\simeq} R_\Delta^\dagger \exp(\pi g(x)).$$

In other words, $\mathcal{M}_\Delta = R_\Delta \exp(\pi g(x))$ *is extended to an isomorphism between* $\widehat{\mathcal{M}_\Delta}$ *and* $R_\Delta^\dagger \exp(\pi g(x))$.

(2) *The isomorphism* $F^! \mathcal{M}_\Delta^{(p)} = R_\Delta^{1/p} \exp(\pi(g^{(p)}(y^p)))$ *is extended to an isomorphism between* $F^! \widehat{\mathcal{M}_\Delta}^{(p)}$ *and* $R_\Delta^{1/p,\dagger} \exp(\pi(g^{(p)}(y^p)))$

Let V_K be the set of O_K-valued polynomial $f(s)$ of $\mathbf{Z}_p^n$ whose total degree on s is not greater than K. Then we have

$$V_K = \oplus_{|i| \le K} \frac{(s)_i}{i!} O_K.$$

Let U_K be the set of R_Δ defined by

$$U_K = \oplus_{w \in (K\Delta) \cup \mathbf{Z}^n} x^w O_K.$$

For a positive rational number ϵ, we put

$$\mathcal{V}(K_0, \epsilon) = \oplus_{K \le K_0} p^{\epsilon K} V_K,$$

$$\mathcal{U}(K_0, \epsilon) = \oplus_{K \le K_0} p^{\epsilon K} U_K.$$

The first statement of Theorem 4.3 is reduced to the following theorem.

Theorem 4.4. *For any sufficiently small number* $\epsilon > 0$, *there exist constants* $A(\epsilon)$ *and* $B(\epsilon)$ *such that*

$$\alpha(\oplus_{i=1}^{\kappa} \mathcal{V}(K_0, \epsilon) N_j \exp(\pi g(x))) \subset A(\epsilon) \mathcal{U}(K_0, \epsilon/2) \exp(\pi g(x))$$

$$\alpha(\oplus_{i=1}^{\kappa} \mathcal{V}(K_0, \epsilon/2) N_j \exp(\pi g(x))) \supset B(\epsilon) \mathcal{U}(K_0, \epsilon) \exp(\pi g(x))$$

for all K_0.

We prove this theorem in the next subsection.

4.2. **Comparison of topologies.** We use the same notations as in the last subsection. We assume that p is good. The main result in this section is the following proposition.

Proposition 4.5. (1) *The image of* $\oplus_{j=1}^{\kappa} V_K N_j$ *under the map* α *is contained in* $U_K = \oplus_{x \in ((K + v(N_j))\Delta) \cup \mathbf{Z}^n} x^w O_K$.

(2) *For any sufficiently small* $\epsilon > 0$, *there exists an element* $A(\epsilon)$ *such that the image of* $\oplus_{j=1}^{\kappa} V_K N_j$ *under the map* α *contains* $A(\epsilon) p^{\epsilon K} U_K = \oplus_{x \in (K\Delta) \cup \mathbf{Z}^n} x^w O_K$.

We use the following lemma for the proof of Proposition 4.5.

Lemma 4.6. (1) *For an element* $\beta = (\beta_1, \ldots, \beta_n) \in \mathbf{Z}_p^n$, *we put* $(\beta s)_k = \prod_{i=1}^{n} (\beta_i s_i)_{k_i}$. *Then* $\frac{(\beta s)_k}{k!}$ *is an element of* $V_{|k|}$ *and* $\frac{(\beta s)_k}{k!} - \beta^k \frac{(s)_k}{k!}$ *is an element of* $V_{|k|-1}$.

(2) *(Formal formula) If f_1, f_2 are expressed formally as $f_1 = g_1\exp(h_1)$, $f_2 = g_2\exp(h_2)$, then we have*

$$\frac{(s)_k}{k!}(f_1\cdot f_2) = \sum_{a+b=k, a,b\geq 0}\frac{(s)_a}{a!}(f_1)\frac{(s)_b}{b!}(f_2)$$

Here for $a = (a_1, \dots, a_n)$, we denote $a \geq 0$ if and only if $a_i \geq 0$ for all i.

Proof. (1) Since $\dfrac{(\beta s)_k}{k!}$ is a $\mathbf{Z}_p$ valued function on $\mathbf{Z}_p^n$, the first statement follows from the definition of $V_{|k|}$. Since the highest degree part of $\dfrac{(\beta s)_k}{k!}$ is equal to $\beta^k s^k$, which is equal to that of $\beta^k\dfrac{(s)_k}{k!}$, the degree of the difference is less than $| k |$.

(2) We can prove the equality by the induction of $| k |$. □

The proof of Proposition 4.5(1). We prove the first statement by the induction on K. The statement is trivial for $K = 0$. We assume the statement for $K - 1$. Let $k \in \mathbf{N}^n$ such that $| k |= K$. Let $b_1, \dots, b_N$ be elements of $\mathbf{N}^n$ such that $\sum_{j=1}^N b_j = k$. We write $g(x) = \sum_{j=1}^N a_j M_j$, with $M_j = x^{w_j}$ and $w_j = (w_{j1}, \dots, w_{jn})$. Put $\beta^{(j)} = (w_{j1}^{-1}, \dots, w_{jn}^{-1})$. Then we have

$$\frac{(\beta^{(j)}s)_{b_j}}{(b_j)!}\exp(\pi a_j x^{w_j}) = \frac{(\pi a_j x^{w_j})^{|b_j|}}{(b_j)!}\exp(\pi a_j x^{w_i}).$$

Therefore the first statmement of Lemma 4.6,

$$\begin{aligned}
&\frac{(s)_{b_j}}{(b_j)!}\exp(\pi a_j x^{w_j}) - w_j^{b_j}\frac{(\beta^{(j)}s)_{b_j}}{(b_j)!}\exp(\pi a_j x^{w_j}) \\
&= \frac{(s)_{b_j}}{(b_j)!}\exp(\pi a_j x^{w_j}) - w_j^{b_j}\frac{(\pi a_j x^{w_j})^{|b_j|}}{(b_j)!}\exp(\pi a_j x^{w_i}) \\
&= \frac{(s)_{b_j}}{(b_j)!}\exp(\pi a_j x^{w_j}) - \frac{\prod_{i=1}^n(\pi a_j w_{ji}x^{w_{ji}})^{b_{ji}}}{(b_j)!}\exp(\pi a_j x^{w_i})
\end{aligned}$$

is contained in $\alpha(V_{|b_i|-1}\exp(\pi a_j M_j))$ and it is contained in $U_{|b_i|-1}\exp(\pi a_j M_j)$ by the hypotheses of the induction. By the equality

$$\sum_{b_1+\cdots+b_N=k}\prod_{j=1}^N\frac{\prod_{i=1}^n(\pi a_j w_{ji}x^{w_{ji}})^{b_{ji}}}{(b_j)!} = \prod_{i=1}^n\frac{(\pi E_i(g))^{k_i}}{(k_i)!}$$

and the second statement of Lemma 4.6,

$$\frac{(s)_k}{k!}\exp(\pi g(x)) - \prod_{i=1}^n\frac{(\pi E_i(g))^{k_i}}{(k_i)!}\exp(\pi g(x))$$

is contained in $U'_{|k|-1}\exp(\pi g(x))$. Therefore

$$\begin{aligned}
&\frac{(s)_k}{k!}N_j\exp(\pi g(x)) - \prod_{i=1}^{n}\frac{(\pi E_i(g))^{k_i}}{(k_i)!}N_j\exp(\pi g(x)) \\
&= N_j\frac{(s-\nu_j)_k}{k!}\exp(\pi g(x)) - \prod_{i=1}^{n}\frac{(\pi E_i(g))^{k_i}}{(k_i)!}N_j\exp(\pi g(x)) \\
&= N_j\frac{(s-\nu_j)_k}{k!}\exp(\pi g(x)) - N_j\frac{(s)_k}{k!}\exp(\pi g(x)) \\
&+ N_j\frac{(s)_k}{k!}\exp(\pi g(x)) - \prod_{i=1}^{n}\frac{(\pi E_i(g))^{k_i}}{(k_i)!}N_j\exp(\pi g(x))
\end{aligned}$$

is contained in $N_jU_{|k|-1}\exp(\pi g(x))$. By the induction hypotheses, we have Proposition 4.5 (1). □

For the proof of Proposition 4.5 (2), it is sufficient to prove the following lemma.

Lemma 4.7. *For any sufficiently small $\epsilon > 0$, there exists $A(\epsilon)$ such that*

$$\mid A(\epsilon)p^{\epsilon K}\mid_p < \mid \frac{\pi^K}{k!}\mid_p$$

for any $k \in \mathbf{N}^n$ such that $\mid k \mid = K$.

Proof. By considering the estimation

$$ord_p(\frac{\pi^K}{k!}) = \frac{K}{p-1} - \sum_{i=1}^{n}\sum_{j=1}^{\infty}[\frac{k_i}{p^j}] \leq n\log_p(K),$$

we have the lemma. □

Proof of Main Theorem 4.3 (2).

$$\widehat{\beta} : F^!\widehat{M_\Delta} = R_\Delta^{1/p}\otimes_{R_\Delta}\widehat{\mathcal{M}_\Delta} \to R_\Delta^{1/p,\dagger}\exp(\pi g(y^p))$$

is obtained by tensoring the isomorphism in (1) with $R_\Delta^{1/p}$. Here we used the canonical isomorphism:

$$R_\Delta^{1/p,\dagger} \simeq R_\Delta^{1/p}\otimes_{R_\Delta}R_\Delta^{\dagger}.$$

□

5. Application to character sums

5.1. **Analyticity of proto γ matrices.** In this sectin, we write down the isomorphism $\widehat{\mathcal{M}_\Delta} \to F^!\widehat{\mathcal{M}_\Delta}$ as $\mathcal{A}(s)$ module explicitly.Let A be a representative of $(\mathbf{Z}/p\mathbf{Z})^n$ in $\mathbf{Z}^n$. We define W_A and $\mathcal{W}_A$ as $W_A = \oplus_{w\in A}Ky^w$, $\mathcal{W}_A = W_A\otimes R_\Delta$, respectively. For $\mathcal{A}(s)$ and $K[s]$ modules $\mathcal{L}$ and L, we introduce $\mathcal{A}(t)$ and $K[s]$ module structures on $W_A\otimes_K\mathcal{L}$ and $W_A\otimes_K L$ by the rule: $f(t)(y^w\otimes l) = y^w\otimes(f(ps-w)l)$ for $f(t)\in\mathcal{A}(t)$ or $f(t)\in K[t]$, $w\in A$ and $l\in\mathcal{L}$ or $l\in L$.

Lemma 5.1. (1) *If $\mathcal{L}$ be a free $\mathcal{A}(s)$ module of rank κ with a base $\{N_1,\dots,N_\kappa\}$, then $W_A\otimes_K\mathcal{L}$ is also a free $\mathcal{A}(t)$ module of rank κ generated by $\{\sum_{w\in A}\chi_{p\mathbf{Z}_p+w}(t)(y^w\otimes N_j)\}_j$.*

(2) *If L be a free $K[s]$ module of rank κ with a base $\{N_1,\dots,N_\kappa\}$, then $W_A\otimes_K L$ is a free $K[t]$ module of rank $p^n\kappa$ generated by $\{y^w\otimes N_j\}_{w\in A, j=1,\dots n}$.*

Definition 5.2. *The element $\sum_{w\in A}\chi_{p\mathbf{Z}_p+w}(t)(y^w\otimes N_j)$ is denoted by $N_{j,A}^{(p)}$ for short.*

It is easy to see that we can take A so that $R_\Delta^{1/p}\subset\mathcal{W}_A$. Until the end of this subsection, we fix such a representative A. Under this situation, we have

$$\widehat{\mathcal{M}_\Delta}\overset{\Phi}{\simeq}F^!\widehat{\mathcal{M}_\Delta}^{(p)}=R_\Delta^{1/p}\otimes_{R_\Delta}\widehat{\mathcal{M}_\Delta}^{(p)}\subset\mathcal{W}_A\otimes_{R_\Delta}\widehat{\mathcal{M}_\Delta}^{(p)},$$
$$F^!\mathcal{M}_\Delta^{(p)}=R_\Delta^{1/p}\otimes_{R_\Delta}\mathcal{M}_\Delta^{(p)}\subset\mathcal{W}_A\otimes_{R_\Delta}\mathcal{M}_\Delta^{(p)}.$$

Note that $R_\Delta^\dagger$ is flat over R_Δ. Under the morphism, $\Phi(N_j)$ can be expressed as

$$\Phi(N_j)=\sum_{i=1}^{\kappa}\gamma_{i,j,A}(t)N_{i,A}^{(p)},=\sum\gamma_{i,j,w}(t)(y^w\otimes N_j)$$

where $\gamma_{i,j,A}(t),\gamma_{i,j,w}(t)\in\mathcal{A}(t)$. We claim that the matrix coefficient $\gamma_{i,j,A}(t)$ is nothing but the proto γ matrix of Dwork [D] if g is homogeneous, i,e, the corresponding $\mathcal{D}_\Delta$ module has only regular singularity. Therefore the matrix $(\gamma_{i,j,A}(t))_{ij}$ is the proto γ function for a irregular $\mathcal{D}_\Delta$ module.To compute the matrix element $\gamma_{i,j,A}(t)$, we use the duality. Note that the original idea of this computation is due to Dwork [D]. We define the module $\mathcal{O}_{-t}^{(p)*}$ by

$$\mathcal{O}_{-t}^{(p)*}=K[t][y_1^{\pm p},\dots,y_n^{\pm p}]_f\cdot y^{-t}/R_{\Delta^c,f}[t]\cdot y^{-t},$$

where

$$K[t][y_1^{\pm p},\dots,y_n^{\pm p}]_f\cdot y^{-t}=\{\sum_{w\in\mathbf{Z}^n}a_wy^{pw}\cdot y^{-t}(\text{ formal sum })\mid a_w\in K[t]\},$$
$$R_{\Delta^c,f}[t]\cdot y^{-t}=\{\sum_{w\in\mathbf{Z}^n-(-C(\Delta))}a_wy^{pw}\cdot y^{-t}(\text{ formal sum })\mid a_w\in K[t]\}.$$

Then $F_i-p\pi g_i^{(p)}(y^p)$ acts on the space $W_{-A}\otimes\mathcal{O}_{-t}^{(p)*}=\oplus_{\kappa\in A}y^{-w}\mathcal{O}_{-t}^{(p)*}$. The $K[t]$ module $\mathcal{K}_A^{(p)}=\cap_iKer((F_i-p\pi g_i^{(p)}(y^p)):W_{-A}\otimes\mathcal{O}_{-t}^{(p)*})$ is a free $K[t]$ module of rank $p^n\kappa$ generated by $\{\xi_{w,j}^*(a,t,y)\}$, where

$$\xi_{w,j}^*(a,t,y)=\sum_u G_j(a^p,\frac{t+w}{p},u)(y^{-w}\otimes y^{-pu})\cdot y^{-t}$$

and $G(a,s,u)$ is defined in §3.3. Moreover under the natrual perfect pairing

$$\mathcal{W}_A\otimes_{R_\Delta}\mathcal{M}_\Delta^{(p)}\times\mathcal{K}_A^{(p)}\to K[t],$$

$\{\xi_{w,j}^*(a,t,y)\}$ is the dual base of $\{y^wN_j(y^p)\}$ of $\mathcal{W}_A\otimes_{R_\Delta}\mathcal{M}^{(p)}$. Let $\sum_{w\in A}y^w\otimes f_w(y^p)\cdot y^t\in W_A\otimes\mathcal{M}^{(p)}$. It is expressed as $\sum_{w\in A,j}a_{w,j}(t)y^w\otimes N_j(y^p)\cdot y^t$

with $a_{w,j}(t) \in \chi_{p\mathbf{Z}_p+w}(t)\mathcal{A}(t)$. The coefficient $a_{w,j}(t)$ is expressed as $a_{w,j}(t) = \chi_{p\mathbf{Z}_p+w}(t)(\sum_{v\in A} y^v \otimes f_v(y^p), \xi^*_{w,j})$. As a consequence, we have the following theorem.

Theorem 5.3. *Under the above notation, the matrix element of proto-γ matrix is given by $\gamma_{A,i,j,w}(t) = \chi_{p\mathbf{Z}_p+w}(\exp(\pi(g(y)-g^{(p)}(y^p)))N_i(y^p), \xi^*_{w,j})$ More explicitly, if $t \in p\mathbf{Z}_p - w$, then*

$$\gamma_{i,j,w} = \sum_{u\in C(\Delta)} c_{p(u-\nu_i)+w}(g)G_j(a^p, \frac{t+w}{p}, u)$$

and it is analytic on $s' \in \mathbf{Z}_p$ where $t = ps' - w$.

5.2. **A generalization of Gross-Koblitz formula.** The following lemma is easy to see.

Lemma 5.4. *Let $u \in (-C(\Delta)^0) \cup \mathbf{Z}^n$. Then (1) $u^iC(\Delta) \supset C(\Delta)$ for $i \geq 0$.(2) $\mathbf{Z}^n = \cup_{i=0}^{\infty} u^iC(\Delta)$.*

The homomorphism $\mathcal{O}_s \to \mathcal{O}_{s+u} : f(x)x^s \mapsto f(x)x^{-u} \cdot x^{s+u}$ indeces a homomorphism

$$\oplus K[s]N_jx^s \simeq \mathcal{O}_s/Im(E-\pi g_i)_i \to \mathcal{O}_{s+u}/Im(E-\pi g_i)_i \simeq \oplus K[s]N_jx^{s+u}.$$

Then we have the following commutative diagram:

Proposition 5.5. *There exist linear functions $L_i = \sum_j a_{ij}s_j + a_{i0}$ with $a_{ij} \in \mathbf{Z}$ and $a_{i0} \in \mathbf{Q}$ such that if $s \in K^n$ and $L_i(s) \neq 0$, then the map*

$$i_u(s) : \oplus KN_jx^s \to \oplus KN_jx^{s+u}$$

is an isomorphism.

Proposition 5.6. *Let $K[s]_{loc} = K[s, \dfrac{1}{L_i(s+v)}]_{v\in\mathbf{Z}^n}$. Then $i_u \otimes K[s]_{loc}$ is an isomorphism.*

Let $L_i^{(1)} = \sum_{i=1}^n a_{ij}s_j$, $R_i^{(p)} = \{-p^k a_{i,0}(\mod \mathbf{Z}) \mid k \in \mathbf{Z}\}$.

Definition 5.7 (p-non-resonsance)**.** *An element $s \in (\mathbf{Q}\cap\mathbf{Z}_p)^n$ is p-resonanant if and only if $L_i(s)(\mod \mathbf{Z}) \notin R_i^{(p)}$.*

The $\mathcal{A}(s)$-linear isomorphism $\Phi : \widehat{\mathcal{M}_\Delta} \to F^!\widehat{\mathcal{M}_\Delta}^{(p)}$ induces $\mathcal{A}(s)_{loc} = \mathcal{A}(s) \otimes_{K[s]} K[s]_{loc}$ linear isomorphism $\Phi : \widehat{\mathcal{M}_{loc}} \to F^!\widehat{\mathcal{M}_{loc}}^{(p)}$ where $\widehat{\mathcal{M}_{loc}} = \widehat{\mathcal{M}_\Delta} \otimes_{K[s]} K[s]_{loc}$ and $F^!\widehat{\mathcal{M}_{loc}}^{(p)} = F^!\widehat{\mathcal{M}_\Delta}^{(p)} \otimes_{K[s]} K[s]_{loc}$. By considering this morphism for $\exp(\pi g^{(p^{i-1})}(y_{(0)}))$, we get the morphism

$$\widehat{\mathcal{M}_{loc}}^{(p^{i-1})} \to F^!\widehat{\mathcal{M}_{loc}}^{(p^i)}.$$

The proto γ-matrix for $\widehat{\mathcal{M}_{loc}}^{(p^{i-1})} \to F^!\widehat{\mathcal{M}_{loc}}^{(p^i)}$ is denoted by $\gamma_{ijw}^{(i-1)}(s_{i-1})$. Apply $(F^{i-1})^!$ to the both side of this isomorphism, we get the following isomorphism

$$\Phi^{(i-1)} : (F^{i-1})^!\widehat{\mathcal{M}_{loc}}^{(p^{i-1})} \to (F^i)^!\widehat{\mathcal{M}_{loc}}^{(p^i)}.$$

We use new variable $y_{(i)}=(y_{(i)1},\dots,y_{(i)n})$ such that $y_{(i)k}^{p^i}=y_{(0)k}$. We evaluate the composite morphism

$$\Phi^{(i-1)}\circ\cdots\circ\Phi^{(0)}:\widehat{\mathcal{M}_{loc}}\to(F^i)^!\widehat{\mathcal{M}_{loc}}^{(p^i)}.$$

of $\mathcal{A}(s_{(0)})_{loc}$ modules at $s_{(0)}=s^0_{(0)}$. We use the following expression of $(F^i)^!\widehat{\mathcal{M}_{loc}}^{(p^i)}$:

$$(F^i)^!\widehat{\mathcal{M}_{loc}}^{(p^i)}=\oplus_{w_1,\dots,w_i\in A,j=1,\dots,\kappa}\chi_{p^i\mathbf{Z}_p^n-(p^{i-1}w_i+\cdots+pw_2+w_1)}\mathcal{A}(s_{(0)})_{loc}$$
$$y_{(i)}^{p^{i-1}w_i+\cdots+pw_2+w_1}\otimes N_j(y_{(i)}^{p^i})\cdot y_{(i)}^{s_{(0)}}.$$

We use the following formula for the mapping $\Phi^{(i)}$:

$$\Phi^{(0)}(N_i(y_{(0)})y_{(0)}^{s_{(0)}})=\sum_{j=1,\dots,\kappa,w_1\in A}\gamma^{(0)}_{ijw_1}(s_{(0)})y_{(1)}^{w_1}\otimes N_j(y_{(1)}^p)\cdot y_{(1)}^{s_{(0)}}$$
$$\cdots$$
$$\Phi^{(k-1)}(N_i(y_{(k-1)}^{p^{k-1}})y_{(k-1)}^{p^{k-1}s_{(k-1)}})=\sum_{j=1,\dots,\kappa,w_k\in A}\gamma^{(k-1)}_{ijw_k}(s_{(k-1)})y_{(k)}^{p^{k-1}w_k}\otimes N_j(y_{(k)}^{p^k})\cdot y_{(k)}^{p^{k-1}s_{(k-1)}}.$$

Then the coefficients of $\gamma^{(0)}_{ijw_1}(s_{(0)}),\dots,\gamma^{(k-1)}_{ijw_k}(s_{(k-1)})$ are elements of $\chi_{p\mathbf{Z}_p-w_1}\mathcal{A}(s_{(0)}),\dots,\chi_{p\mathbf{Z}_p-w_k}\mathcal{A}(s_{(k-1)})$.

We specialize $s_{(0)}$ to $s^0_{(0)}$ and compute $\Phi^{(f-1)}\circ\cdots\circ\Phi^{(0)}(N_i(y_{(0)})y_{(0)}^{s^0_{(0)}})$. Then we have

(5.1)
$$\Phi^{(1)}\circ\Phi^{(0)}(N_i(y_{(0)})y_{(0)}^{s^0_{(0)}})=\Phi^{(1)}(\sum_{j=1,\dots,\kappa,w_1\in A}\gamma^{(0)}_{ijw_1}(s^0_{(0)})y_{(1)}^{w_1}\otimes N_j(y_{(1)}^p)\cdot y_{(1)}^{s^0_{(0)}})$$

Since the sum for w_1 is zero except for $w_1+s^0_{(0)}\in p\mathbf{Z}_p^n$, by putting $s^0_{(1)}=\dfrac{w_1+s^0_{(0)}}{p}$, the (5.1) is equal to

$$\Phi^{(1)}(\sum_{j=1,\dots,\kappa}\gamma^{(0)}_{ijw_1}(s^0_{(0)})1\otimes N_j(y_{(1)}^p)\cdot y_{(1)}^{ps^0_{(1)}})$$
$$=\sum_{j,k=1,\dots,\kappa,w_2\in A}\gamma^{(0)}_{ijw_1}(s^0_{(0)})\gamma^{(0)}_{jkw_2}(s^0_{(1)})y_{(2)}^{pw_2}\otimes N_j(y_{(2)^{p^2}})\cdot y_{(2)}^{ps^0_{(1)}}$$
$$=\sum_{j,k=1,\dots,\kappa}\gamma^{(0)}_{ijw_1}(s^0_{(0)})\gamma^{(0)}_{jkw_2}(s^0_{(1)})y_{(2)}^{pw_2+w_1}\otimes N_j(y_{(2)^{p^2}})\cdot y_{(2)}^{s^0_{(0)}}$$

For the last expression, we choose $w_2 \in A$ such that $w_2 + s^0_{(1)} \in p\mathbf{Z}_p^n$. In the same way, we define w_k and $s^0_{(k)}$ inductively by the relation

$$w_k + s^0_{(k-1)} \in p\mathbf{Z}_p^n, s^0_{(k)} = \frac{w_k + s^0_{(k-1)}}{p},$$

and we have

$$\begin{aligned} &\Phi^{(k-1)} \circ \cdots \circ \Phi^{(0)}(N_i(y_{(0)}) \cdot y_{(0)}^{s^0_{(0)}}) \\ = &\sum_{j_1,\ldots,j_{k-1}=1,\ldots,\kappa} \gamma^{(0)}_{ij_1w_1}(s^0_{(0)}) \cdots \gamma^{(k-1)}_{j_{k-1}j_kw_k}(s^0_{(k-1)}) \\ &y_{(k)}^{p^{k-1}w_k+\cdots+pw_2+w_1} \otimes N_{j_k}(y^{p^k}_{(k)}) \cdot y_{(k)}^{s^0_{(0)}}. \end{aligned}$$

Let f be a natural number and $q = p^f$. Assume that $(q-1)s^0_{(0)} = p^{f-1}w_f + \cdots + pw_2 + w_1$. Then we have

$$y_{(f)}^{p^{f-1}w_k+\cdots+pw_2+w_1} \otimes N_{j_f}(y^{p^f}_{(f)}) \cdot y_{(f)}^{s^0_{(0)}} = 1 \otimes N_{j_f}(y^q_{(f)}) \cdot y_{(f)}^{qs^0_{(0)}}.$$

We put $\Gamma^{(k-1)}_{w_k} = (\gamma^{(k-1)}_{ijw_k})_{ij}$. Then we have

$$\begin{pmatrix} N_1(y_{(0)})y_{(0)}^{s^0_{(0)}} \\ \vdots \\ N_\kappa(y_{(0)})y_{(0)}^{s^0_{(0)}} \end{pmatrix} = \Gamma^{(0)}_{w_1}(s^0_{(0)}) \cdots \Gamma^{(f-1)}_{w_f}(s^0_{(f-1)}) \begin{pmatrix} N_1(y^q_{(f-1)})y_{(f-1)}^{qs^0_{(0)}} \\ \vdots \\ N_\kappa(y^q_{(f-1)})y_{(f-1)}^{qs^0_{(0)}} \end{pmatrix}$$

Theorem 5.8 (Boyarski Principle)**.** *Let $a_i \in \mu_{q-1}$ and $\delta(a_i) \neq 0$ mod p. Under the above notation, we have*

$$tr(\Gamma^{(0)}_{w_1}(s^0_{(0)}) \cdots \Gamma^{(f-1)}_{w_f}(s^0_{(f-1)})) = \sum_{x \in \mu^n_{q-1}} \psi(tr\bar{g}(x)))x^u,$$

where $u = (q-1)s^0_{(0)} = p^{f-1}w_f + \cdots + pw_2 + w_1$, $\bar{g}(x) = g(x)(\text{ mod } p)$ and ψ is the additive character of $\mathbf{F}_q$ defined by θ.

Proof. We use the trace formula due to Dwork. □

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