FUNDAMENTAL GROUPS OF MODULI SPACES OF HYPERPLANE CONFIGURATIONS

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§0 Introduction

To study the monodromy representation for hypergeometric functions, it is a fundamental problem to determine the fundamental group of the moduli space on which the hypergeometric functions are defined. For example, the fundamental group of the moduli space of distinct ordered n points in $\mathbf{C}$ acts on the space of Appel hypergeometric functions. This fundamental group is called the pure braid group. There is a natural generalizaiton of this hypergeometric function called Aomoto-Gel'fand hypergeometric function. This is defined as the integral of the product of the complex powers of linear forms on an affine space. In this direction, Aomoto and Matsumoto-Sasaki-Takayama-Yoshida gave the monodromy transfromation for certain paths ([A], [MSTY]). To study the monodromy action, it is natural problem to determine the structure of the fundamental group of the moduli space of normal crossing hyperplane configurations in $\mathbf{C}^n$.

In this paper, we give a set of generators and relations of the fundamental group of the configuraion space $X(n,k)$ of normal crossing k hyperplanes in $\mathbf{P}^{n-1}$ in combinatorial language (Main Theorem 5.7.). Let me explain the strategy of computing $\pi_1(X(n,k))$. First we take very special base point $p = p(\lambda_1, \dots, \lambda_k)$ in $X(n,k)$ called Vandermonde point depending on a sequence of real numbers $\lambda_1 > \dots > \lambda_k > 0$. To describe all the data of configurations in terms of combinatorial language, we introduce an order on the rational function field $\mathbf{Q}(\lambda_1, \dots, \lambda_k)$ such that this order coincide with the usual order of $\mathbf{R}$ for a finite set of $\mathbf{Q}(\lambda_1, \dots, \lambda_k)$ if $\lambda_1, \dots, \lambda_k$ are specialized to sufficiently good real numbers. Next we consider embeddings $\mu(i) : \mathbf{C} \to (\hat{\mathbf{P}}^{n-1})^k$ to get sufficiently many elements in $\pi_1(X(n,k))$. The most of this paper is spent to get the exact relations between these elements. Roughly speaking, there are two parts to get enough relations. First we compute the fundamental group of the fiber $C([1,k]-i)$ of the map $X(n,k) \to X(n,k-1)$ obtined by forgetting i-th hyperplane. This description is also obatined by Lawrence ([L]). To get the description of $\pi_1(C([1,k]-i))$, we define the first inclusions $\nu(i) : \mathbf{P}^2 \to (\hat{\mathbf{P}}^{n-1})^k$ to apply Zariski's theorem and use the technic of Randell, Salvetti, Arvola ([R],[S],[AR]). Next we study the relations between these normal subgroups $\pi_1(C([1,k]-i))$ $(i = 1, \dots, k)$ in $\pi_1(X(n,k))$ by considering the second inclusions $\mu(i,j)$ from $\mathbf{C}^2 \to (\hat{\mathbf{P}}^{n-1})^k$. In the last two secitons, we put everything together to prove that a set of generators and relations are enough to describe $\pi_1(X(n,k))$.

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§1 Preliminary

§1.1 A total order for the field $\mathbf{Q}(\lambda_1, \ldots, \lambda_k)$

In this section, we introduce a total order on the rational function field $\mathbf{Q}(\lambda_1, \ldots, \lambda_k)$ of $\lambda_1, \ldots, \lambda_k$ and study its property. For monomials $\lambda_1^{e_1} \cdots \lambda_k^{e_k}$ and $\lambda_1^{f_1} \cdots \lambda_k^{f_k}$, $\lambda_1^{e_1} \cdots \lambda_k^{e_k} \prec \lambda_1^{f_1} \cdots \lambda_k^{f_k}$ if and only if there exists an m $(1 \leq m \leq k)$ such that $e_1 = f_1, \ldots, e_{m-1} = f_{m-1}$ and $e_m < f_m$. In particular, $1 \prec \lambda_n \prec \lambda_{n-1} \prec \cdots \prec \lambda_1$. This gives a total order of the set of the monomials $\{\lambda^M = \lambda_1^{m_1} \cdots \lambda_k^{m_k} \mid M \in \mathbf{N}^k\}$. For a polynomial $f(\lambda) = \sum a_M \lambda^M$, in $\mathbf{Q}[\lambda_1, \ldots, \lambda_k]$, $0 \prec f$ if and only if $a_{M_0} > 0$, where $\lambda^{M_0} = max\{\lambda^M \mid a_M \neq 0\}$ and for two polynomials $f(\lambda), g(\lambda) \in \mathbf{Q}[\lambda_1, \ldots, \lambda_k]$, $f \prec g$ if and only if $0 \prec g - f$. This defines a total order on $\mathbf{Q}[\lambda_1, \ldots, \lambda_k]$.

Proposition 1.1. *(1) The inclusion $\mathbf{Q}[\lambda_1, \ldots, \hat{\lambda}_i, \ldots, \lambda_k] \subset \mathbf{Q}[\lambda_1, \ldots, \lambda_k]$ is order perserving.*

(2) For polynomials $f_1, \ldots, f_m \in \mathbf{Q}[\lambda_1, \ldots, \lambda_k]$ such that $f_1 \prec \cdots \prec f_m$, there exist $\lambda_i \in \mathbf{R}$ such that $f_1(\lambda) < \cdots < f_m(\lambda)$.

Proof. (1) It is clear.

(2) We show that $\lambda_n = \exp(k), \ldots, \lambda_1 = \exp(k^n)$ for a suffciently large k satisfies the required inequalities. For such $\lambda_1, \ldots, \lambda_n$, $\lambda^M = \exp(m_1 k^n + \cdots + m_n k)$. So if $\lambda^{M_1} \prec \lambda^{M_2}$ then $\lambda^{M_1}/\lambda^{M_2} \geq \exp(k)$ for a sufficiently large k. If we write $f_2 - f_1 = a_{M_0}\lambda^{M_0} + \sum_{\lambda_M < \lambda_{M_0}} a_M \lambda^M$ with $a_{M_0} > 0$, it is sufficient to choose k such that $\exp(k) a_{M_0} > \sum_{\lambda_M < \lambda_{M_0}} \mid a_M \mid$ to get $f_1(\lambda) < f_2(\lambda)$. In the same way, we can choose a sufficiently large k_i such that $f_i(\lambda^{(i)}) < f_{i+1}(\lambda^{(i)})$ with $\lambda_j^{(i)} = exp(k_i^{n-j+1})$. Then the maximum $k = max_{1 \leq i \leq m-1} k_i$ satisfies the required inequalites.

Using this order, we introduce a total order on the rational function field $\mathbf{Q}(\lambda_1, \ldots, \lambda_k)$ of $\lambda_1, \ldots, \lambda_k$ as follows. $g_1/f_1 \prec g_2/f_2$ if and only if $0 \prec f_1 f_2(f_1 g_2 - f_2 g_1)$. The next corollary is a direct cosequence of Proposition 1.1.

Corollary 1.2. *(1) For rational funcitons $f_1, \ldots, f_m$ in $\mathbf{Q}(\lambda_1, \ldots, \lambda_k)$ such that $f_1 \prec \cdots \prec f_m$, there exist $\lambda_1, \ldots, \lambda_k \in \mathbf{R}$ such that (i) $f_1(\lambda), \ldots, f_m(\lambda)$ are defined and (ii) $f_1(\lambda) < \cdots < f_m(\lambda)$.*

(2) The inclusion $\mathbf{Q}(\lambda_1, \ldots, \hat{\lambda}_i, \ldots \lambda_k) \subset \mathbf{Q}(\lambda_1, \ldots, \lambda_k)$ is order perserving.

Definition. Such $\lambda_1, \ldots, \lambda_k$ are called o-generic for $f_1, \ldots, f_m$. Unless otherwise mentioned, we consider sufficiently o-generic λ_i's and we write $<$ instead of $\prec$.

§1.2 Computation of the fundamental group of the complement of arrangement in dimension 2

In this section, we review the result of Randell, Salvetti and Arvola. To state the result, we introduce a transformation of generators of a free group associated to a line configuration in $\mathbf{C}^2$ which is the complexification of real one.

Let $\pi : \mathbf{C}^2 \ni (x, y) \to x \in \mathbf{C}$ be the first projection and $D = \cup_{i \in I} L_i$ a configuration of lines in $\mathbf{C}^2$ which doesn't have $\pi^{-1}(0)$ and $\{y = 0\}$ as a component of D. Suppose that $\pi^{-1}(0)$ and $\{y = 0\}$ crosses normally to D. A component L_i is called vertical if it is contained in a fiber $\pi^{-1}(\xi)$ of ξ and a component L_i is called horizontal if it is not vertical. Let $I_v = \{i \mid L_i \text{ is vertical }\}$ and $I_b = \{i \mid L_i \text{ is horizontal }\}$. A complex number x is called a singular value if $\#(\pi^{-1}(x) \cap D) \leq \#I_h - 1$ or $\pi^{-1}(x) \subset D$ and generic otherwise. Since the configuration is defined over $\mathbf{R}$, the set of the singular values Σ is a finite subset of $\mathbf{R}$ and $\Sigma \cup 0$ is ordered by the ordering of $\mathbf{R}$: $\Sigma \cup 0 = \{x_1 < \cdots < x_m\}$. Assume that if $(x_i, 0) \in D$ then $\pi^{-1}(x_i)$ is a component of D. Let us choose $\xi_1, \ldots, \xi_{m+1}$ such that $\xi_1 < x_1 < \xi_2 < \cdots < \xi_m < x_m < \xi_{m+1}$ and $(\xi_i, 0) \notin D$. Now we define isomorphisms between fundamental groups $\pi_1(\pi^{-1}(\xi_i) - D, (\xi_i, 0))$. Note that they are free group of rank $\#I_h$. We connect $(\xi_i, 0)$ and $(\xi_{i+1}, 0)$ by a path γ_i defined in the follwing way. The set $S_i = \{(x, 0) \mid \xi_i \leq x \leq \xi_{i+1}\} \cap (\cup_{i \in I_h} L_i)$ is decomposed into S_i^+ and S_i^- where $S_i^\pm = \{x \mid \text{ the component which passes } (x, 0) \text{ is } y = ax + b \text{ with } \pm a > 0\}$. We connect $(\xi_i, 0)$ and $(\xi_{i+1}, 0)$ by a path defined as the union of

(1) $\{(t, 0) \mid t \in \mathbf{R}, \xi_i \leq t \leq \xi_{i+1}, \mid t - x \mid \geq \epsilon \text{ for all } x \in S_i\} \cup \{x_i\}$
(2) $\cup_{x \in S_i^+ \cup \{x_i\}} \{(t, 0) \mid t \in \mathbf{C}, \mid t - x \mid = \epsilon, \Im t \leq 0\}$
(3) $\cup_{x \in S_i^-} \{(t, 0) \mid t \in \mathbf{C}, \mid t - x \mid = \epsilon, \Im t \geq 0\}$

(See figure 1.)

(Figure 1)

This path induces an isomorphism:

$$\gamma_i : \pi_1(\pi^{-1}(\xi_i) - D, (\xi_i, 0)) \overset{\simeq}{\to} \pi_1(\pi^{-1}(\xi_{i+1}) - D, (\xi_{i+1}, 0))$$

Now we are interested in the distinguished generators

$$\gamma_j^{(i)} \in \pi_1(\pi^{-1}(\xi_i) - D, (\xi_i, 0)) \text{ for } j \in I_h,$$

and the relation between them via the isomorphism γ_i. For $j \in I_h$, define a path $\gamma_j^{(i)}$ in $\pi^{-1}(\xi_i) - D$ with the base point $(\xi_i, 0)$ using the complex coordinate y for (ξ_i, y): Connect from 0 to $p_{i,j} + i\epsilon$, where $p_{i,j}$ is the y-cooridinate of $L_j \cap \pi^{-1}(\xi_i)$ and from there go around $p_{i,j}$ anti-clockwise $p_{i,j} + i\epsilon e^{i\theta}$ $(0 \leq \theta \leq 2\pi)$ and go back to 0. (See figure 2.)

(Figure 2)

For $x_i = 0$, $\gamma_j^{(i)}$ is also denoted as γ_j^0. Before stating the relation between $\{\gamma_j^{(i)}\}_{j \in I_h}$ and $\{\gamma_j^{(i+1)}\}_{j \in I_h}$, we introduce the index set $T_i = \{y \mid \text{ The point } (x_i, y) \text{ lies in more than two lines } L_j \ (j \in I_h)\}$ for $i = 1, \ldots, m$ and $U_y = \{j \in I_h \mid L_j \ni (x_i, y)\}$ for $y \in T_i$. For $j, k \in U_y$, we write $j \prec k$ if $a_j < a_k$ where $L_j = \{y = a_j x + b_j\}$ and $L_i = \{y = a_i x + b_i\}$.

Proposition 1.3. *Under the above isomorphism γ_i, for $y \in T_i$, we have*

$$\gamma_{j_t}^{(i)} \cdots \gamma_{j_1}^{(i)} = \gamma_{j_1}^{(i+1)} \cdots \gamma_{j_t}^{(i+1)} \text{ for } (t = 1, \dots, k),$$

where $U_y = \{j_1 \prec \cdots \prec j_k\}$ and $\gamma_j^{(i)} = \gamma_j^{(i+1)}$ if $j \notin \cup_{y \in T_i} U_y$.

By this proposition, $\gamma_j^{(i)}$ can be regarded as an element of the free group generated by γ_i^0. Now we are ready to state the structure of the fundamental group.

Theorem 1.4.(Randell [R], Salvetti [S], Arvola [AR]). *Suppose that $I_v = \emptyset$ and the closure $\bar{D}$ of $D = \cup_{i \in I} L_i$ in $\mathbf{P}^2$ crosses normally to the infinite line. Moreover, assume that the line $\{y = 0\}$ is not pararell to any component of D.*

(1) The fundamental group $\pi_1(\mathbf{C}^2 - \cup_{l \in I} L_l, (0,0))$ is generated by γ_j^0 $(j \in I_h)$ and the relations are given by

$$[\gamma_{j_k}^{(i)} \cdots \gamma_{j_1}^{(i)}, \gamma_{j_l}^{(i)}] = 1 \qquad (l = 1, \dots, k), \tag{1.1}$$

where $x_i \in \Sigma \cup \{0\}, y \in T_i$ and $U_y = \{j_1 \prec \cdots \prec j_k\}$. (2) The fundamental group $\pi_1(\mathbf{P}^2 - \bar{D}, (0,0))$ of $\mathbf{P}^2 - \bar{D}$ is generated by γ_i^0 $(i \in I_h)$ and the relations are given by (1.1) as above and

$$\gamma_{j_1}^0 \cdots \gamma_{j_m}^0 = 1, \tag{1.2}$$

where $I_h = \{j_1, \dots, j_m\}$ and $L_{j_l} \cap \pi^{-1}(0) = (0, y_l)$ with $y_1 < \cdots < y_m$,

Proof. The proof is given in [OT].

Remark. If $y \in T_i$ is normal crossing point, then U_y has only two element say $\{p, q\}$. Then the condition (1.1) is nothing but the commutativity relation of $\gamma_p^{(i)}$ and $\gamma_q^{(i)}$. Therefore $\gamma_p^{(i)} = \gamma_p^{(i+1)}$ for $p \in I_h$ if U_y contains only two elements for all $y \in T_i$.

§2 Moduli spaces and the first inclusion.

§2.1 The moduli space $X(n,k), C(I)$ and the element $\gamma_{i,K}$

Let $n \geq 2, k \geq n+1$ be integers, $\mathbf{P}^{n-1} = \{(x_0 : \cdots : x_{n-1})\}$ and $\hat{\mathbf{P}}^{n-1} = \{(\xi_0 : \cdots : \xi_{n-1})\}$ be the dual projective space of $\mathbf{P}^{n-1}$ where $\xi = (\xi_0 : \cdots : \xi_{n-1})$ corresponds to the hyperplane $\xi_0 x_0 + \cdots + \xi_{n-1} x_{n-1} = 0$ in $\mathbf{P}^{n-1}$. The moduli space $X(n,k)$ of k hyperplanes in $\mathbf{P}^{n-1}$ is the open set of $\hat{\mathbf{P}}^{n-1} \times \cdots \times \hat{\mathbf{P}}^{n-1}$ defined by $X(n,k) = \{(\xi^{(1)}, \dots, \xi^{(k)}) \in (\hat{\mathbf{P}}^{n-1})^k \mid \cup_{i=1}^k \xi^{(i)}$ is a normal crossing divisor$\}$. For a subset K in $[1,k] = \{1, \dots, k\}$ such that $\#K = n$, $det(\xi^{(i)})_{i \in K} = 0$ defines a divisor D_K in $(\hat{\mathbf{P}}^{n-1})^k$. $X(n,k)$ is equal to the complement of the union $\cup_{\#K=n, K \subset [1,k]} D_K$ of divisors D_K. To study the fundamental group of $X(n,k)$, we also consider the special point $p \in X(n,k)$ and the space $C(I)$ for a subset $I \subset [1,k]$ as follows. Let $\xi(\lambda)$ be the point in $\hat{\mathbf{P}}^{n-1}$ given by $(1 : (-\lambda) : \cdots : (-\lambda)^{n-1})$ and $\lambda_i \in \mathbf{C}$ $(i = 1, \dots, k)$ be complex numbers such that $\lambda_i \neq \lambda_j$ if $i \neq j$. Then $p = p(\lambda_1, \dots, \lambda_k) = (\xi(\lambda_1), \dots, \xi(\lambda_k))$ is a point of $(\hat{\mathbf{P}}^{n-1})^k$ and it is called Vandermonde point. Since $det(\xi(\lambda_i))_{i \in K} = \pm \prod_{i<j, \in K} (\lambda_i - \lambda_j) \neq 0$, this lies in $X(n,k)$.

Let $I = \{i_1, \ldots, i_m\}$ be a subset of $[1, k]$ such that $\#I \geq n-1$. Define an open set $C(I)$ of $\hat{\mathbf{P}}^{n-1}$ by

$$C(I) = \{\xi \in \hat{\mathbf{P}}^{n-1} \mid \xi \cup \cup_{i \in I} \xi(\lambda_i) \text{ is a normal crossing divisor of } \mathbf{P}^{n-1}\}.$$

By definition, the fiber of the map from $X(n, k)$ to $X(n, k-1)$ for $k \geq n$ defined by

$$X(n, k) \ni (\xi^{(1)}, \ldots, \xi^{(k)}) \mapsto (\xi^{(1)}, \ldots, \hat{\xi}^{(i)}, \ldots, \xi^{(k)}) \in X(n, k-1)$$

at $(\xi(\lambda_1), \ldots, \hat{\xi}(\lambda_i), \ldots, \xi(\lambda_k))$ is equal to $C([1, k] - i)$. Here we used the notation $[1, k] - i$ for $[1, k] - \{i\}$.

Proposition 2.1. *Let I be a subset of $[1, k]$ such that $\#I \geq n-1$. Then $C(I)$ is the complement of $\cup_{K \subset I, \#K = n-1} L_K$ in $\hat{\mathbf{P}}^{n-1}$, where L_K is a hyperplane defined by*

$$l_K : D_K^{(n-1)} \xi_0 + D_K^{(n-2)} \xi_1 + \cdots + D_K^{(0)} \xi_{n-1} = 0,$$

and $D_K^{(i)}$ is the i-th elementary symmetric polynomial in $\{\lambda_j\}_{j \in K}$.

Proof. Since the complement of $C(I)$ is the union of hyperplanes L_K ($\#K = n-1, K \subset I$) defined by $L_K : l_k = \det(\xi, \xi^{(i)})_{i \in K} = 0$. This linear form on ξ is characterized by $l_K(1, (-\lambda_i), \ldots, (-\lambda_i)^{n-1}) = 0$ for $i \in K$ up to constant. By the equality $D_K^{(n-1)} + D_K^{(n-2)}(-x) + \cdots + D_K^{(0)}(-x)^{n-1} = \prod_{i \in K}(\lambda_i - x)$, this equation satisfies this property.

From now on, $\mathcal{A}(i)$ denotes the index set $\{K \subset [1, k] - i \mid \#K = n-1\}$. Define a map $\mu(i)$ form $\mathbf{C}$ to $(\hat{\mathbf{P}}^{n-1})^k$ by

$$x \mapsto (\xi(\lambda_1), \ldots, \xi(\lambda_i, x), \ldots, \xi(\lambda_k)),$$

where $\xi(\lambda_i, x) = (1 : (-\lambda_i) : \cdots : (-\lambda_i)^{n-1} + x) \in \hat{\mathbf{P}}^{n-1}$. Then we have the following propostion.

Proposition 2.2. *The set $\mu(i)^{-1}(X(n, k))$ is the complement of $\{p_K = -\prod_{j \in K}(\lambda_j - \lambda_i)\}_{K \in \mathcal{A}(i)}$.*

Proof. Since the map $\mu(i)$ factors through $\mu(i)^{-1}(X(n, k)) \to C([1, k] - i) \to X(n, k)$, the corresponding complemente is defined by the equation

$$D_K^{(n-1)} 1 + D_K^{(n-2)}(-\lambda_i) + \cdots + D_K^{(0)}((-\lambda_i)^{n-1} + x) = 0$$

for $K \in \mathcal{A}(i)$. This equation is nothing but $\prod_{j \in K}(\lambda_j - \lambda_i) + x = 0$, so we have the lemma.

Now we define elements $\gamma_{i,K} \in \pi_1(X(n, k), p)$ for $(i \in [1, k], K \in \mathcal{A}(i))$ as follows. Here p denotes the Vandermonde points $(\xi(\lambda_1), \ldots, \xi(\lambda_k))$. Using Proposition 2.2, we have a map of pointed variety:

$$\mu(i) : (\mathbf{C} - \{p_K\}_{K \in \mathcal{A}(i)}, 0) \to (X(n, k), p),$$

and it induces a homomorphism of the fundamental groups:

$$\mu(i)_* : \pi_1(\mathbf{C} - \{p_K\}_{K\in\mathcal{A}(i)}, 0) \to \pi_1(X(n,k), p).$$

To give a specified element in $\pi_1(\mathbf{C}-\{p_K\}_{K\in\mathcal{A}(i)}, 0)$, we assume that $\lambda_1, \ldots, \lambda_k \in \mathbf{R}$ are o-generic for $\{p_K\}_{K\in\mathcal{A}(i)}$.

Definition ($\gamma_{i,K}$). The loop $\tilde{\gamma}_{i,K}$ is defined as connecting 0 to $p + i\epsilon$ and rounding p_k anti-clockwise $p + i\epsilon e^{i\theta}$ $(0 \le \theta \le 2\pi)$ and going back to 0. (See figure 3.)

(Figure 3)

The image of $\tilde{\gamma}_{i,K}$ in the group $\pi_1(X(n,k), p)$ under the homomorphism $\mu(i)_*$ is denote by $\gamma_{i,K}$.

§2.2 The first inclusions and their genericity.

From §2.2 to §3.2, we obtain sufficiently many relations satisfied by $\gamma_{i,K}$ $(i \in [1,k], K \in \mathcal{A}(i))$ defined in §2.1. In the rest of this section, we compute the fundamental group of $C([1,k]-i)$.

Definition ($\nu(i)$). For $i = 1, \ldots, k$, we define an inclusion $\nu(i)$ from $\mathbf{P}^2$ to $\hat{\mathbf{P}}^{n-1}$ by

$$(t : x : y) \mapsto (t : (-\lambda_i)t : \cdots : (-\lambda_i)^{n-2}t + x : (-\lambda_i)^{n-1}t + y).$$

By inhomogenizing, this morphism is written as

$$(x, y) \mapsto (1 : (-\lambda_i) : \cdots : (-\lambda_i)^{n-2} + x : (-\lambda_i)^{n-1} + y).$$

By the description of Proposition 2.1, we get the following proposition.

Proposition 2.3. *For $K \in \mathcal{A}(i)$, the inverse image $L'_K = \nu(i)^{-1}(L_K)$ of L_K by $\nu(i)$ is defined by the equation:*

$$(*) \qquad L'_K : l'_K = \prod_{j\in K}(\lambda_j - \lambda_i) + (\sum_{j\in K}\lambda_j)x + y = 0$$

Now we study the induced configuration $D' = \nu(i)^{-1}(\cup_{K\in\mathcal{A}(i)}L_K)$ in $\mathbf{P}^2$. By the above proposition, the codimention 1 strata in $\mathbf{P}^2$ are parametrized by $\mathcal{A}(i)$. For the codimention 2 strata, we have the following proposition.

Proposition 2.4. *(1) The codimention 2 strata of D' are parametrized as follows.*

(1) *Non-normal crossing points are parametrized by $\mathcal{B}(i) = \{L \subset [1,k] - i \mid \#L = n-2\}$.*
(2) *Normal crossing point is parametrized by the set of unordered pairs $\{K_1, K_2\}$ of $K_1, K_2 \in \mathcal{A}(i)$ with $\#(K_1 \cap K_2) < n-2$.*

(2) There is one to one correspondence between the set of codimension 2-strata of D' and that of codimension 2-strata of $D \subset \mathbf{P}^{n-1}$ and the image $\nu(i)(\mathbf{P}^2)$ doesn't meet any codimension 3 stratum.

Proof. (1) Let $K_1, K_2 \in \mathcal{A}(i)$ and $\# K_1 \cap K_2 = n-2$. If we write $L = K_1 \cap K_2$ and $K_1 = L \cup \{p\}$, the value of l'_{K_1} at $(x,y) = (-\prod_{j \in L}(\lambda_j - \lambda_i), (\sum_{j\in L}\lambda_j + \lambda_i)\prod_{j\in L}(\lambda_j - \lambda_i))$ is equal to

$$(\lambda_p - \lambda_i)\prod_{j\in L}(\lambda_j - \lambda_i) - (\sum_{j\in L}\lambda_j + \lambda_p)\prod_{j\in L}(\lambda_j - \lambda_i) + (\sum_{j\in L}\lambda_j + \lambda_i)\prod_{j\in L}(\lambda_j - \lambda_i) = 0$$

So the intersection point $L'_{K_1} \cup L'_{K_2}$ depends only on $K_1 \cap K_2$. Therefore to prove the proposition, it is enough to prove that if $\#(K_1 \cap K_2) < n-2$, the pair of indices K_1, K_2 is recovered from the coordinates of the intersection $L'_{K_1} \cap L'_{K_2}$. From the equation (*), the intersection is computed as

$$(x,y) = (-\frac{D_1 - D_2}{T_1 - T_2}, \frac{T_1 D_2 - T_2 D_1}{T_1 - T_2}),$$

where $D_p = \prod_{j \in K_p}(\lambda_j - \lambda_i)$ and $T_p = \sum_{j\in K_p}\lambda_j$ for $p = 1, 2$. We put $\xi_{K_1,K_2} = -\frac{D_1 - D_2}{T_1 - T_2}$.

Lemma 2.5. *If $\xi_{K_1,K_2} \in \mathbf{Q}[\lambda_1, \dots, \lambda_k]$, then $\#(K_1 \cap K_2) = n-2$.*

Proof. Let $p \in K_1 - K_2$ and take the partial differential of $-\xi_{K_1,K_2}(T_1 - T_2) = D_1 - D_2$, with respect to λ_p. Since $deg_{\lambda_p}(T_1 - T_2) = deg_{\lambda_p}(D_1 - D_2) = 1$, $deg_{\lambda_p}\xi_{K_1,K_2} = 0$. We have $\xi_{K_1,K_2} = \frac{\partial}{\partial \lambda_p}(D_1 - D_2) = \prod_{j \in K_1 - p}(\lambda_j - \lambda_i)$. In the same way, if we take $q \in K_2 - K_1$, we have $\xi_{K_1,K_2} = \prod_{j\in K_2 - q}(\lambda_j - \lambda_i)$. So we have $K_1 - p = K_2 - q$.

The proof of (1). By this lemma the denominator $T_1 - T_2$ of ξ_{K_1,K_2} is recovered up to $\mathbf{Q}^\times$ multiple if $\#(K_1 \cap K_2) < n-2$. By multiplying $T_1 - T_2$, we can recover $D_1 - D_2$ form ξ_{K_1,K_2} up to $\mathbf{Q}^\times$ multiple. The constant term of $D_1 - D_2$ with respect to λ_i is $\prod_{j\in K_1}\lambda_j - \prod_{j \in K_2}\lambda_j$, so we can recover K_1 and K_2 up to permutation.

(2). For $K_1, K_2 \in \mathcal{A}(i)$ with $\#(K_1 \cap K_2) < n-2$ and $K_3 \neq K_1, K_2 \in \mathcal{A}(i)$, we have $L'_{K_1} \cap L'_{K_2} \cap L'_{K_3} = \emptyset$. Therefore it is enough to see that $\cap_{L \subset K} L_K$ has codimension 2 in $\hat{\mathbf{P}}^{n-2}$ if $L \in \mathcal{B}(i)$. Since $\cap_{L\subset K} L_K$ contains the codimension 2 linear subspace generated by $(1 : (-\lambda_i) : \cdots : (-\lambda_i)^{n-1})$ $(j \in L)$, its codimension is 2.

Corollary 2.6. *If $\lambda_1, \dots, \lambda_k$ are sufficiently o-generic, the induced homomorphism*

$$\nu(i)_* : \pi_1(\nu(i)^{-1}(C([1,k] - i)), (0,0)) \to \pi_1(C([1,k]-i), p)$$

induced by $\nu(i)$ is an isomorphism.

proof. Since the image of this inclusion doesn't meet codimension 3-stratum, we have the isomorphism by Zariski's theorem.

§2.3 The fundamental group of $C(I)$.

Now we compute the fundamental group of $\nu(i)^{-1}(C([1,k]-i)$ according to the method explained in §1. We take the affine piece of $\mathbf{P}^2$ as the corrdinate (x,y) given as above and consider the first projection $\pi : \mathbf{C}^2 \ni (x,y) \mapsto x \in \mathbf{C}$. This coordinate satisfies the condition in the begining of §1.2 and there is no vertical components in the configuration $D' = \nu(i)^{-1}(\cup_{K\in\mathcal{A}(i)} L_K)$. First we consider the relation arising from non-normal crossing points.

Definition ($\mathcal{B}(i)^*, \mathcal{A}_L(i)$). (1) Since the set $\mathcal{B}(i)^* = \mathcal{B}(i) \cup \{0\}$ can be identified with the subset $\{-\prod_{j\in L}(\lambda_j - \lambda_i)\} \cup \{0\}$, the order in $\mathbf{Q}(\lambda_1, \dots, \lambda_k)$ induces an order on $\mathcal{B}(i)^*$. For an element $L \in \mathcal{B}(i)^*$, the minimum element in $\mathcal{B}(i)^*$ greater than L is denoted by L^+.

(2) For an element $L \in \mathcal{B}(i)$, define $\mathcal{A}_L(i) = \{K \in \mathcal{A}(i) \mid L \subset K\}$. This set can be identified with the subset $\{-\prod_{j\in K}\lambda_j\}_{K\in\mathcal{A}_L(i)}$ of $\mathbf{Q}(\lambda_1, \dots, \lambda_k)$. Therefore there is an induced order $<$ on $\mathcal{A}_L(i)$.

Definition($\gamma_K^{(L)}$). We define elements $\gamma_K^{(L)} \in \pi_1(\pi^{-1}(0) \cap \nu(i)^{-1}(C([1,k]-i)), (0,0))$ by the following relations :

(1) $\gamma_K^{(0)} = \gamma_K^{(0^+)} = \gamma_{i,K}$ $(K \in \mathcal{A}(i))$
(2) $\gamma_{K_t}^{(L)} \cdots \gamma_{K_1}^{(L)} = \gamma_{K_1}^{(L^+)} \cdots \gamma_{K_t}^{(L^+)}$ for $t = 1, \dots, k-n+1$, where $\mathcal{A}_L(i) = \{K_1 < \cdots < K_{k-n+1}\}$ and $\gamma_K^{(L)} = \gamma_K^{(L^+)}$ if $K \in \mathcal{A}(i) - \mathcal{A}_L(i)$.

To describe the commutation relation arising from normal crossing point corresponding to $K_1, K_2 \in \mathcal{A}(i)$ with $\#(K_1 \cap K_2) < n-2$, define $L(K_1, K_2)$ as the minimal $L \in \mathcal{B}(i)^*$ such that

$$-\frac{D_1 - D_2}{T_1 - T_2} < -\prod_{j\in L}(\lambda_j - \lambda_i),$$

where D_1, D_2, T_1 and T_2 is defined in §2.2.

Theorem 2.7. *Assume that $\lambda_1, \dots, \lambda_k$ are sufficiently o-generic. Then the fundamental group of $C([1,k]-i)$ is generated by $\gamma_K^{(0)} = \gamma_{i,K}$ and the relations are given by*

$$[\gamma_K^{(L)}, \gamma_{K_{k-n+1}}^{(L)} \cdots \gamma_{K_1}^{(L)}] = 1 \tag{2.1}$$

for $K \in \mathcal{A}_L(i) = \{K_1 < \cdots < K_{k-n+1}\}$

$$[\gamma_{K_1}^{(L(K_1,K_2))}, \gamma_{K_2}^{(L(K_1,K_2))}] = 1 \tag{2.2}$$

for $K_1, K_2 \in \mathcal{A}(i)$ and $\#(K_1 \cap K_2) < n-2$

$$\gamma_{K_1}^{(0)} \cdots \gamma_{K_p}^{(0)} = 1 \tag{2.3}$$

$$\text{where } \mathcal{A}(i) = \{K_1, \dots, K_p\} \text{ with } -\prod_{j\in K_1}(\lambda_j - \lambda_i) < \cdots < -\prod_{j\in K_p}(\lambda_j - \lambda_i).$$

Proof. The set of singular value for the projection $\mathbf{C}^2 \ni (x,y) \mapsto x \in \mathbf{C}$ is $\{-\prod_{i\in L}(\lambda_l - \lambda_i) \mid L \in \mathcal{B}(i)\} \cup \{-\frac{D_1-D_2}{T_1-T_2} \mid K_1, K_2 \in \mathcal{A}(i), \#(K_1 \cap K_2) < n-2\}$ and degree 1 coefficients of the equaitons of the lines l_K is $-\prod_{l\in K}\lambda_l$. The lines l_K cross with $\{x=0\}$ at the points $\{-\prod_{i\in K}(\lambda_l - \lambda_i) \mid K \in \mathcal{A}(i)\}$. Since $\lambda_1, \dots, \lambda_k$ are o-generic, the definition of $\gamma_K^{(L)}$ is nothing but the element which we constructed in §1.2. Therefore we get the theorem from Theorem 1.4.

§3 The second inclusion.

§3.1 The second inclusion and its genericity.

In §2.3, we computed the fundamental group of $C([1,k]-i)$ for fixed i. In this section, we study relations between the fundamental groups of $C([1,k]-i)$ and $C([1,k]-j)$ embedded in that of $X(n,k)$. First we introduce an inclusion $\mu(i,j)$ $(i,j \in [1,k], i \neq j)$.

Definition ($\mu(i,j)$). For $i,j \in [1,k]$, we define the map $\mu(i,j)$ from $\mathbf{C}^2$ to $(\hat{\mathbf{P}}^{n-1})^k$ by

$$\mathbf{C}^2 \ni (x,y) \mapsto (\xi(\lambda_1), \dots, \xi(\lambda_i, x), \dots, \xi(\lambda_j, y), \dots, \xi(\lambda_k)) \in (\hat{\mathbf{P}}^{n-1})^k,$$

where $\xi(\lambda, x) = (1 : (-\lambda) : \cdots : (-\lambda)^{n-1} + x)$.

First we study the configuration $D' = \mu(i,j)^{-1}(\cup_{\#K=n} D_K)$ in $\mathbf{C}^2$, where D_K is the divisor defined in §2.1.

Proposition 3.1. *The codimension 1 strata of D' consist of 3 kinds of lines :*

(1) Lines pararell to $x=0$ are parametrized by $\mathcal{A}(i,j) = \{K \subset [1,k]-i-j \mid \#K = n-1\}$. The equation of the line $l_K^{(1)}$ corresponding to $K \in \mathcal{A}(i,j)$ is given by

$$l_K^{(1)} : x = -\prod_{l\in K}(\lambda_l - \lambda_i).$$

(2) Lines pararell to $y=0$ are parametrized by $K \in \mathcal{A}(i,j)$. The equation of the line $l_K^{(2)}$ which corresponds to K is:

$$l_K^{(2)} : y = -\prod_{l\in K}(\lambda_l - \lambda_j).$$

(3) Lines not pararell to $x=0$ nor $y=0$ pararmetrized by $\mathcal{B}(i,j) = \{L \subset [1,k]-i-j \mid \#L = n-2\}$. Then the equation of l_L is given by

$$l_L : \frac{x}{(\lambda_j - \lambda_i)\prod_{l\in L}(\lambda_l - \lambda_i)} + \frac{y}{(\lambda_i - \lambda_j)\prod_{l\in L}(\lambda_l - \lambda_j)} = -1.$$

Proof. $\mu^{-1}(D_K)$ is non empty if and olny if $K \cap \{i,j\} \neq \emptyset$. We show that according to $K \cap \{i,j\} = \{i\}$,$\{j\}$ or $\{i,j\}$, $\mu^{-1}(D_K)$ is given by (1), (2) or (3).

(i) The case $K \cap \{i,j\} = \{i\}$. Since the condition $det((1 : (-\lambda_i) : \cdots : (-\lambda_i)^{n-1} + x), \xi(\lambda_l))_{l \in K - \{i\}}$ dosen't depend on y, and $\mu(i,j)(0,x) = \mu(i)(x)$ defined in §2. Therefore we have $\mu(i,j)^{-1}(D_K) = \{x = p_{K-\{i\}}\}$, where $p_{K-\{i\}} = -\prod_{l \in K - \{i\}} (\lambda_l - \lambda_i)$.

(ii) The case $K \cap \{i,j\} = \{j\}$. This is similar to the case (i).

(iii) The case $K \ni i,j$. If $\mu(i,j)(x,0) \cap D_K \neq \emptyset$, the by the same reason, we have $x = -\prod_{l \in K-\{i\}}(\lambda_l - \lambda_i)$. Therefore we have the required equation with $L = K - \{i,j\}$.

As for the non-normal crossing point, we have the following proposition.

Proposition 3.2. *The non-normal crossing points of D' are parametrized by $\mathcal{A}$ $(i,j) = \{K \in [1,k] - i - j \mid \#K = n-1\}$ and the coordinate of the point p_K correspontding to K is given by $p_K = (-\prod_{k \in K}(\lambda_k - \lambda_i), -\prod_{k \in K}(\lambda_k - \lambda_j))$. Moreover the lines which pass through this point are $l_K^{(1)}$, $l_K^{(2)}$ and l_L ($L \in \mathcal{B}_K = \{K \in \mathcal{B}(i,j) \mid L \subset K, \#L = n-2\}$).*

Proof. It is easy to see that p_K lies in $l_K^{(1)}$, $l_K^{(2)}$ and l_L ($L \in \mathcal{B}_K$) by the direct computation. Since the coordinate of the intersection of $l_K^{(1)}$ and l_L is

$$(-\prod_{k \in K}(\lambda_k - \lambda_i), (\lambda_j - \lambda_i)\prod_{l \in L}(\lambda_l - \lambda_j) - \frac{\prod_{l \in L}(\lambda_l - \lambda_j)}{\prod_{l \in L}(\lambda_l - \lambda_i)} \prod_{k \in K}(\lambda_k - \lambda_i)),$$

y-coordinate is not a polynomial in λ_i if $L \not\subseteq K$. Therefore it is enough to prove the following proposition.

Proposition 3.3. *(1) If $L_1, L_2 \in \mathcal{B}(i,j)$ and $\#L_1 \cap L_2 < n-3$, then the x-coordintate and y-coordinate of the intersection l_{L_1} and l_{L_2} are not polynomial in λ_i and these coordinates recover the indices L_1 and L_2 up to permutation.*

(2) If $L \not\subseteq K$, then the y-coordinate of the intersection of $l_K^{(1)}$ and l_L recovers the index L.

Proof. (1) The y-coordinate of $l_{L_1} \cap l_{L_2}$ is given by

$$y = \frac{\prod_{k \in L_1}(\lambda_k - \lambda_i)\prod_{k \in L_2}(\lambda_k - \lambda_i)\{\prod_{k \in L_1}(\lambda_j - \lambda_k) - \prod_{k \in L_2}(\lambda_j - \lambda_k)\}}{\prod_{k \in L_1}(\lambda_i - \lambda_k)\prod_{k \in L_2}(\lambda_j - \lambda_k) - \prod_{k \in L_2}(\lambda_i - \lambda_k)\prod_{k \in L_1}(\lambda_j - \lambda_k)}$$

Lemma 3.4. *If $\#A \geq 2$ or $\#B \geq 2$ and $A \cap B = A \cap \{i,j\} = B \cap \{i,j\} = \emptyset$, then the polynomial*

$$\frac{1}{\lambda_i - \lambda_j}\{\prod_{a \in A}(\lambda_i - \lambda_a)\prod_{b \in B}(\lambda_j - \lambda_b) - \prod_{b \in B}(\lambda_i - \lambda_b)\prod_{a \in A}(\lambda_j - \lambda_a)\}$$

is irreducible.

Proof. We prove this by the induction of $\#A + \#B$. It is easy to see in the case where $\#A = 2$ and $\#B = 1$ by direct computation. Since the argument is symmetric in

A and B, we may assume that there exists an $\alpha \in A$ such that $\#(A-\alpha) \geq 2$ or $\#B \geq 2$. We assume the lemma for $A-\alpha$ and B. Notice that the given polynomial is degree 1 polynomial with respect to λ_α. The coefficient of λ_α is

$$(*) \qquad \frac{1}{\lambda_i-\lambda_j}\{\prod_{a\in A-\alpha}(\lambda_i-\lambda_a)\prod_{b\in B}(\lambda_j-\lambda_b)-\prod_{b\in B}(\lambda_i-\lambda_b)\prod_{a\in A-\alpha}(\lambda_j-\lambda_a)\}$$

and it is irreducible by the assumption of the induction. The constant term with respect to λ_α is

$$\frac{1}{\lambda_i-\lambda_j}\{\lambda_i\prod_{a\in A-\alpha}(\lambda_i-\lambda_a)\prod_{b\in B}(\lambda_j-\lambda_b)-\lambda_j\prod_{b\in B}(\lambda_i-\lambda_b)\prod_{a\in A-\alpha}(\lambda_j-\lambda_a)\}$$

which is clearly not a multiple of (*). So it is irreducible.

To prove the statement (1), we prove that from y-coordinate

$$(**) \quad y=\frac{\prod_{k\in L_1^0}(\lambda_k-\lambda_i)\prod_{k\in L_2^0}(\lambda_k-\lambda_i)\{\prod_{k\in L_1}(\lambda_j-\lambda_k)-\prod_{k\in L_2}(\lambda_j-\lambda_k)\}}{\prod_{k\in L_1^0}(\lambda_i-\lambda_k)\prod_{k\in L_2^0}(\lambda_j-\lambda_k)-\prod_{k\in L_2^0}(\lambda_i-\lambda_k)\prod_{k\in L_1^0}(\lambda_j-\lambda_k)},$$

where $L_1^0=L_1-L_1\cap L_2$ and $L_2^0=L_2-L_1\cap L_2$, we can recover L_1 and L_2. If we put $u=\#L_1^0=\#L_2^0$, then the denominator of (**) is $(\lambda_i-\lambda_j)\times$ (irreducible polynomial of degree $2u-1$). On the other hand nominator is $\prod$ (linear form) $\times$ (degree u polynomial) . So if $u\geq 2$, then there is a unique irreducible factor up to $\mathbf{Q}^\times$ in denominator of degree $2u-1$ say $F(\lambda)$. Therefore y-coordinate is not a polynomial of λ_i. We can prove the statement for x-coordinate in the same way. As a polynomial of λ_i,λ_j the coefficient of $\lambda_i^u\lambda_j^0$ in $(\lambda_i-\lambda_j)F(\lambda)$ is equal to $\prod_{p\in L_1^0}\lambda_p-\prod_{q\in L_2^0}\lambda_q$. So the y-coordinate recovers L_1^0 and L_2^0 up to permutation. By the description (**), we have

$$y(\lambda_i-\lambda_j)F(\lambda_k)=\prod_{k\in L_1\cup L_2}(\lambda_k-\lambda_i)(\prod_{k\in L_1^0}(\lambda_j-\lambda_k)-\prod_{k\in L_2^0}(\lambda_j-\lambda_k))$$

and by this we can recover $L_1\cup L_2$.

(2) The y-coordinate of the intersection $l_K^{(1)}\cap l_L$ is given by

$$\begin{aligned} y&=(\lambda_j-\lambda_i)\prod_{l\in L}(\lambda_l-\lambda_j)-\frac{\prod_{l\in L}(\lambda_l-\lambda_j)}{\prod_{l\in L}(\lambda_l-\lambda_i)}\prod_{k\in K}(\lambda_k-\lambda_i).\\ &=\frac{\prod_{l\in L}(\lambda_l-\lambda_j)}{\prod_{l\in L^0}(\lambda_l-\lambda_i)}(\prod_{l\in L^0\cup j}(\lambda_l-\lambda_i)-\prod_{l\in K^0}(\lambda_l-\lambda_i)),\end{aligned}$$

where $L^0=L-K$ and $K^0=K-L$. The polynomial $\prod_{l\in L^0\cup j}(\lambda_l-\lambda_i)-\prod_{l\in K^0}(\lambda_l-\lambda_i)$ is irreducible and degree is $\#L^0+1$ because the constant term with respect to λ_i is irreducible. Therefore if $\#L^0>0$, then the y-coordinate recovers L^0 and K^0. Therefore it also recovers L and K.

Using this second inclusion $\mu(i,j) : \mathbf{C}^2 \to (\hat{\mathbf{P}}^{n-1})^k$, we obtain (1) the equality between some conjugate of $\gamma_{i,L\cup j}$ and that of $\gamma_{j,L\cup i}$ and (2) the action of $\gamma_{j,L} \in \pi_1(C([1,k]-j),p)$ on $\pi_1(C([1,k]-i),p)$. To get these relation, we define another generators $\{\gamma_{i,K_2}^{(K_1)}\}_{K_2\in\mathcal{A}(i)}$ and $\{\gamma_{j,K}^*\}_{K\in\mathcal{A}(j)}$ of $\pi_1(C([1,k]-i),p)$ and $\pi_1(C([1,k]-j),p)$ respectively.

Definition ($\mathcal{B}(i,j)^{\pm}$, $\gamma_{i,K}^*$, $\gamma_{j,M}^{(K)}$). (1) The subset of $\mathcal{B}(i,j)$ consisting of $L \in \mathcal{B}(i,j)$ such that $\prod_{l\in L}(\lambda_l-\lambda_i)\prod_{l\in L}(\lambda_l-\lambda_j) > 0$ (resp. < 0) is denoted by $\mathcal{B}(i,j)^+$ (resp. $\mathcal{B}(i,j)^-$). Note that the set $D' \cap \{y=0\}$ is indexed by the union $\mathcal{A}(i,j)\cup\mathcal{B}(i,j)$. Here, the index sets $\mathcal{A}(i,j)$ and $\mathcal{B}(i,j)$ correspond to the intersection with $\{y=0\}$ and vertical divisors and horizontal divisors respectively. By this correspondence, $\mathcal{B}(i,j)^+$ and $\mathcal{B}(i,j)^-$ correspond to the union of S_i^+ and that of S_i^- respectively under the notation in §1.2. Since the set $\{-\prod_{l\in K}(\lambda_l-\lambda_i)\}_{K\in\mathcal{A}(i,j)}\cup\{-\prod_{l\in L}(\lambda_l-\lambda_i)(\lambda_j-\lambda_i)\}_{L\in\mathcal{B}(i,j)}$ can be identified with the set $\mathcal{A}(i,j)\cup\mathcal{B}(i,j)$, there is an induced order on $\mathcal{A}(i,j)\cup\mathcal{B}(i,j)$. Attaching an elements $L\in\mathcal{B}(i,j)$ with $L\cup\{j\}$, one can identify $\mathcal{A}(i,j)\cup\mathcal{B}(i,j)$ with $\mathcal{A}(i)$. By this identification, $\mathcal{A}(i)$ is also ordered.

(2) We define the element $\gamma_{i,K}^*$ for $K\in\mathcal{A}(i)$ in $\pi_1(\mu(i)^{-1}(X(n,k)),(0,0))$ which is isomorphic to the free group generated by $\{\gamma_{i,K} \mid K\in\mathcal{A}(i)\}$ as follows.

(1) If $K>0$, then

$$\gamma_{i,K}^*\gamma_{i,K_1}\cdots\gamma_{i,K_p} = \gamma_{i,K_1}\cdots\gamma_{i,K_p}\gamma_{i,K},$$

where $\{K' \mid 0 < K' < K \text{ and } K'\in\mathcal{A}(i,j) \text{ or } K = j\cup L \text{ with } L\in\mathcal{B}(i,j)^+\} = \{K_1<\cdots<K_p\}$.

(2) If $K<0$, then

$$\gamma_{i,K_1}\cdots\gamma_{i,K_p}\gamma_{i,K}^* = \gamma_{i,K}\gamma_{i,K_1}\cdots\gamma_{i,K_p},$$

where $\{K' \mid K < K' < 0 \text{ and } K'\in\mathcal{A}(i,j) \text{ or } K = j\cup L \text{ with } L\in\mathcal{B}(i,j)^+\} = \{K_1<\cdots<K_p\}$.

The path $\gamma_{i,K}^*$ corresponds to the path connecting $(0,0)$ and $(-\prod_{l\in K}(\lambda_l-\lambda_i)-\epsilon,0)$ by the method explained in §1.2 and going around small circle anti-clockwisely and going back to $(0,0)$ via the same path. (See figure 4.)

(Figure 4)

(3) We introduce another order on the set $\mathcal{A}(i,j)\cup\{0\}$, using the order of $\{-\prod_{l\in K}(\lambda_l-\lambda_j)\}\cup\{0\}$. We define an element $\gamma_{j,M}^{(K)}$ for $K\in\mathcal{A}(i,j)$ and $M\in\mathcal{A}(j)$ in the group $\pi_1(\mu(j)^{-1}(X(n,k)),(0,0))$ which is isomorphic to the free group generated by $\{\gamma_{j,M} \mid M\in\mathcal{A}(j)\}$ in the following way. First we introduce a subset $\mathcal{A}_K(j) = \{L\cup i \mid L\subset K, \#L = n-2\}\cup\{K\}$ of $\mathcal{A}(j)$. It can be identified with the subset $\{\frac{\prod_{l\in L}(\lambda_l-\lambda_j)}{\prod_{l\in L}(\lambda_l-\lambda_i)}\}\cup\{0\}$ of $\mathbf{Q}(\lambda_1,\ldots,\lambda_k)$ and $\mathcal{A}_K(j)$ has an induced order. This order coincides with the order of linear coefficient of lines which pass through

non-normal crossing point p_K. We write $\mathcal{A}_K(j)=\{M_1<\cdots<M_n\}$. Then $\gamma_{j,M}^{(K)}$ is defined by

$$\gamma_{j,M}=\gamma_{j,M}^{(0)}$$

$$\gamma_{j,M_t}^{(K)}\cdots\gamma_{j,M_1}^{(K)}=\gamma_{j,M_1}^{(K^+)}\cdots\gamma_{j,M_t}^{(K^+)},\ \text{for } t=1,\dots,n$$

$$\gamma_{j,M}^{(K)}=\gamma_{j,M}^{(K+)}\ \text{if } M\in\mathcal{A}(j)-\mathcal{A}_K(j),$$

where $K^+\in\mathcal{A}(i,j)$ is the smallest element greater than K. The path $\gamma_{j,M}^{(K)}$ corresp-nds to a path in $\pi_1(\mu(i,j)^{-1}(X(n,k))\cap\{x=p_K-\epsilon\},(p_K-\epsilon,0))$ for a sufficiently small ϵ defined as in §1.2 via the path connecting $(0,0)$ and $(p_K-\epsilon,0)$. (See also figure 4.)

By Proposition 1.3, we get the action of $\gamma_{i,K}^*$ on the group $<\gamma_{j,M}>$.

Theorem 3.5. *Assume that $\lambda_1,\dots,\lambda_k$ are sufficiently o-generic. Then we have the following relation in $\pi_1(X(n,k),p)$. (1) For $L\in\mathcal{B}(i,j)$, we have the relation:*

$$\gamma_{i,L\cup j}^*=\gamma_{j,L\cup i}^{([L\cup j])},\tag{3.1}$$

where $[L\cup j]$ denotes the maximal element K of $\mathcal{A}(i,j)$ such that $\prod_{l\in K}(\lambda_l-\lambda_i)>\prod_{l\in L\cup j}(\lambda_l-\lambda_i)$. Since $\gamma_{i,L\cup j}^*$ is conjugate to the element $\gamma_{i,L\cup j}$, $\gamma_{i,L\cup j}$ is contained in the normal subgroup generated by $\gamma_{j,K}$.

(2) For $K\in\mathcal{A}(i,j)$, we have

(3.2)
$$\gamma_{i,K}^*(\gamma_{j,M_t}^{(K)}\cdots\gamma_{j,M_1}^{(K)})\gamma_{i,K}^{*\,-1}=(\gamma_{j,M_n}^{(K)}\cdots\gamma_{j,M_1}^{(K)})^{-1}\gamma_{j,M_t}^{(K)}\cdots\gamma_{j,M_1}^{(K)}(\gamma_{j,M_n}^{(K)}\cdots\gamma_{j,M_1}^{(K)})$$

where $\mathcal{A}_K(j)=\{M_1<\cdots<M_n\}$ and

$$[\gamma_{j,M}^{(K)},\gamma_{i,K}^*]=1\ \textit{if } M\in\mathcal{A}(j)-\mathcal{A}_K(j).\tag{3.3}$$

Definition ($G(n,k)$). Let $G(n,k)$ be the group defined by generators $\gamma_{i,K}$ ($K\in\mathcal{A}(i)$) and relations (2.1), (2.2), (2.3), (3.1), (3.2) and (3.3).

Corollary 3.6. *(1) The subgroup of $G(n,k)$ generated by $\{\gamma_{j,K}\mid K\in\mathcal{A}(j)\}$ is a normal subgroup in $G(n,k)$.*

(2) In the group $G(n,k)$, the element $\gamma_{i,L\cup j}$ ($L\in\mathcal{B}(i,j)$) is contained in the subgroup generated by $\{\gamma_{j,K}\mid K\in\mathcal{A}(j)\}$.

(3) $G(n,k)/<\gamma_{k,K}>$ is isomorphic to $G(n,k-1)$.

Proof. (1) Since $<\gamma_{i,K}^*>_{K\in\mathcal{A}(i)}=<\gamma_{i,K}>_{K\in\mathcal{A}(i)}$, $<\gamma_{j,M}^{(K)}>_{M\in\mathcal{A}(j)}=<\gamma_{j,M}>_{M\in\mathcal{A}(j)}$ and the relations of (3.1), (3.2) and (3.3) in Theorem 3.5 are satisfied, we get (1).

(2) Since $\gamma_{i,L\cup j}^*$ is conjugate to $\gamma_{i,L\cup j}$, it is contained in the normal subgroup generated by $\gamma_{j,K}$ ($K\in\mathcal{A}(j)$) by (3.1) of Theorem 3.5. On the other hand, $<\gamma_{j,K}>$ is a normal subgroup, we get the theorem.

(3) By (2), $G(n,k)/<\gamma_{k,K}>$ is generated by $\gamma_{i,K}$ with $k\notin K$ and the defining relation is (2.1), (2.2), (2.3), (3.1), (3.2) and (3.3) modulo $<\gamma_{k.K}>$. Since $\gamma_{i,K}^{(L)}$

$(K \in \mathcal{A}(i))$, $\gamma_{j,M}^{(K)}$ $(K \in \mathcal{A}(i,j), M \in \mathcal{A}(j))$ and $\gamma_{i,K}^{*}$ $(K \in \mathcal{A}(i))$ are conjugate to $\gamma_{i,K}$, $\gamma_{j,M}$ and $\gamma_{i,K}$ respectively, the meaningful relation of the form (2.1) occurs only when $k \notin L$, otherwise $L \subset K$ implies $\gamma_{i,K}^{(L)} = 1 \mod < \gamma_{k,K} >$. We claim that in case $k \notin L$, the relation (2.1) mod $< \gamma_{i,K} >_{k \in K}$ for K is nothing but the relation (2.1) for $(k-1)$. Let $\bar{\mathcal{A}}(i)$ and $\bar{\mathcal{B}}(i)$ be index sets $\{K \in \mathcal{A}(i) \mid k \notin K\}$ and $\{L \in \mathcal{B}(i) \mid k \notin L\}$ respectively and $R_K^{(L)}$ be a word of $\bar{\gamma}_{i,K}$ $(K \in \bar{\mathcal{A}}(i))$ defined by the same method as the relation $r_K^{(L)}$ for $(k-1)$ in the left hand side of (2.1).

$$R_K^{(L)} = [\bar{\gamma}_K^{(L)}, \bar{\gamma}_{\bar{K}_{k-n}}^{(L)} \cdots \bar{\gamma}_{\bar{K}_1}^{(L)}], r_K^{(L)} = [\gamma_K^{(L)}, \gamma_{K_{k-n+1}}^{(L)} \cdots \gamma_{K_1}^{(L)}]$$

$$\text{for } K \in \bar{\mathcal{A}}_L(i) = \bar{\mathcal{A}}(i) \cap \mathcal{A}_L(i) = \{\bar{K}_1 < \cdots < \bar{K}_{k-n}\},$$

$$\mathcal{A}_L(i) = \{K_1 < \cdots < K_{k-n+1}\}$$

It is enough to prove $R_K^{(L)} = r_K^{(L)} \mod < \gamma_{i,K} >_{k \in K}$ for $L \in \bar{\mathcal{B}}(i)$ to show the claim. Since the inclusion $\bar{\mathcal{B}}(i) \subset \mathcal{B}(i)$ is order preserving by Proposition 1.1, and by the definition, we can prove this equality by induction, noticing that the defining relation (2.1) of $\gamma_K^{(L)}$ reduces to trivial if $L \in \mathcal{B}(i) - \bar{\mathcal{B}}(i)$. In the same way, we can prove the similar claim for the relations (2.2), (2.3), (3.1), (3.2) and (3.3). Therefore we have $G(n,k)/ < \gamma_{k,K} > \simeq G(n,k-1)$.

§4 Falk divisor for the map $X(n,k) \to X(n,k-1)$

§4.1 Divisor Z_{K_1,K_2,K_3}

Let us assume that $k \geq n+2$. The map $X(n,k) \to X(n,k-1)$ obtained by sending $(\xi^{(1)}, \ldots, \xi^{(k)})$ to $(\xi^{(1)}, \ldots, \xi^{(k-1)})$ is called the fogetfull map and denoted by $f : X(n,k) \to X(n,k-1)$. As is claimed in [F] and [M], this map fails to be a fibration in general, therefore we investigate the strucure of f to get some informations about the relation between $\pi_1(X(n,k))$ and $\pi_1(X(n,k-1))$. In this paragraph, we study the combinatorial description for the bad divisor for the map f.

Definition (Manin-Schechtman lattice). Let D_K be the divisor of $(\hat{\mathbf{P}}^{n-1})^k$ defined in §2.1. A pair $(\xi^{(1)}, \ldots, \xi^{(k)})$ in $X(n,k-1)$ is said to belong the Manin-Schechtman lattice if the set of divisors $\{L_K = D_K \cap f^{-1}(\xi^{(1)}, \ldots, \xi^{(k-1)}) \mid K \in \mathcal{A}(k)\}$ in $\hat{\mathbf{P}}^{n-1}$ satisfies the following condition:

(*) For $K_1, K_2, K_3 \in \mathcal{A}(k)$, $codim(L_{K_1} \cap L_{K_2} \cap L_{K_3}) = 2$ if and only if $\#(K_1 \cap K_2 \cap K_3) = n-2$.

Note that this condition is up to codimension 2 condition for the lattice of configuration $\{L_K\}_{K \in \mathcal{A}(k)}$. The set of points which belong to the Manin-Schechtman lattice forms an open set of $X(n,k-1)$ and will be denoted by $X(n,k-1)_{MS}$. There is also a largest open set of $X(n,k-1)$ such that the intersection lattice of the fiber is constant. This open set is called the smooth part and denoted by

$X(n,k-1)_{SM}$. It is easy to see that $X(n,k-1)_{SM} \subset X(n,k-1)_{MS} \subset X(n,k-1)$. The closed set $B(n,k-1)_{MS} = X(n,k-1) - X(n,k-1)_{MS}$ is called the Falk subvariety. The codimension 1 component of $B(n,k-1)$ is called the Falk divisor.

Proposition 4.1. *The components of Falk divisor are indexed by unordered triple* $\{K_1, K_2, K_3\}$ *where* $K_1, K_2, K_3 \in \mathcal{A}(k) = \{K \subset [1,k-1] \mid \#K = n-1\}$ *such that* $K_1 \cap K_2 = K_2 \cap K_3 = K_3 \cap K_1$ *and its cardinality is* $n-3$. *Moreover the component corresponding to* $\{K_1, K_2, K_3\}$ *is defined as*

$$Z_{K_1,K_2,K_3} = \{(\xi^{(1)}, \ldots, \xi^{(k-1)}) \mid codim(L_{K_1} \cap L_{K_2} \cap L_{K_3}) = 2\}$$

Proof. Let $K_1, K_2, K_3 \in \mathcal{A}(k)$ such that $\#M \leq n-3$ where $M = K_1 \cap K_2 \cap K_3$.

Lemma 4.2. *Under the above notation, if* $Z_{K_1,K_2,K_3} \neq \emptyset$, *then* $K_1 \cap K_2 = K_2 \cap K_3 = K_3 \cap K_1$.

Proof. For a sebset A of $[1,k-1]$, L_A denotes the linear hull of $\{\xi^{(i)}\}_{i \in A}$. This notation is compatible with the notation already defined for $K \in \mathcal{A}(k)$. If $K_1 \cap K_2 = M' \supsetneq M = K_1 \cap K_2 \cap K_3$, take $m \in M' - M$. Since the codimension of $L_{K_1} \cap L_{K_2}$ is 2 and by the condition of Z_{K_1,K_2,K_3}, $codim(L_{K_1} \cap L_{K_2} \cap L_{K_3}) = 2$, we have $L_{K_3} \supset L_{K_1} \cap L_{K_2} \cap L_{K_3} = L_{K_1} \cap L_{K_2} \supset L_{M'} \ni \xi^{(m)}$. This contradicts to the generic condition and $m \notin K_3$.

Let $\#M = n-m-1$, where $M = K_1 \cap K_2 \cap K_3$. We count the codimension of Z_{K_1,K_2,K_3}. For a subset A of $[1,k-1]$, $\mathbf{P}^A$ denotes $(\hat{\mathbf{P}}^{n-1})^{\#A} = \{(\xi_i)_{i \in A} \mid \xi_i \in \hat{\mathbf{P}}^{n-1}\}$. Since the subvariety Z_{K_1,K_2,K_3} is the pull back of a subvariety $Z^*_{K_1,K_2,K_3}$ in $\mathbf{P}^{\tilde{M}}$ where $\tilde{M} = K_1 \cup K_2 \cup K_3$. The codimension of Z_{K_1,K_2,K_3} is equal to that of $Z^*_{K_1,K_2,K_3}$. Here the subvariety $Z^*_{K_1,K_2,K_3}$ is defined by

$$Z^*_{K_1,K_2,K_3} = \{(\xi^{(i)})_{i \in \tilde{M}} \mid codim(L_{K_1} \cap L_{K_2} \cap L_{K_3}) = 2\}.$$

The fiber of the natural projection $Z^*_{K_1,K_2,K_3} \to \mathbf{P}^{K_2 \cup K_3}$ at the generic point $(\tilde{\xi}^{(i)})$ is identified with $X = \{(\xi^{(i)})_{K_1 - M} \mid$ the linear hull of $(\xi^{(i)})_{K_1}$ contains $L_{K_2} \cap L_{K_3}\}$. Since the map from $\{(\xi^{(i)})_{K_1-M} \mid (\xi^{(i)}, \tilde{\xi}^{(j)})_{j \in M}$ is normal crossing $\}$ to the Grassman variety $Gr(n-2 : \hat{\mathbf{P}}^{n-1}; L_M)$ of hyperplanes in $\hat{\mathbf{P}}^{n-1}$ containing L_M attaching L_{K_1} is surjective and X is isomorphic to the inverse image of the varieties $Gr(n-2 : \hat{\mathbf{P}}^{n-1}; L_{K_1} \cap L_{K_2})$ corresponding to hyperplanes which contain $L_{K_1} \cap L_{K_2}$. Since $Gr(n-2 : \hat{\mathbf{P}}^{n-1}; L_M)$ and $Gr(n-2 : \hat{\mathbf{P}}^{n-1}; L_{K_1} \cap L_{K_2})$ are m and 1 dimension respectively, therefore the codimension is $m-1$.

So far, we proved the following fact,

(1)
$$B(n,k-1)_{MS} = \cup_{K_1 \cap K_2 = K_2 \cap K_3 = K_3 \cap K_1} Z_{K_1,K_2,K_3}$$

(2)
$$codim(Z_{K_1,K_2,K_3}) = n-2-\#(K_1 \cap K_2 \cap K_3)$$

To prove the Proposition 4.1, it is enough to prove the following lemma.

Lemma 4.3. *If the divisor Z_{K_1,K_2,K_3} is equal to $Z_{K'_1,K'_2,K'_3}$, then $\{K_1,K_2,K_3\}=\{K'_1,K'_2,K'_3\}$.*

Proof. We put $M=K_1\cap K_2\cap K_3$. To prove the lemma, it is enough to prove the following lemma.

Lemma 4.4. *(1) Let $a\in K_1-M$. Then for a generic $(\tilde{\xi}^{(i)})_{i\neq a}$, the subvariety $\{\xi\in\hat{\mathbf{P}}^{n-1}\mid(\xi,\tilde{\xi}^{(i)})_{i\neq a}\in Z_{K_1,K_2,K_3}\}$ is a hyperplane in $\hat{\mathbf{P}}^{n-1}$.*

(2) Let $m\in M$. Then for a generic $(\tilde{\xi}^{(i)})_{i\neq m}$, the subvariety $\{\xi\in\hat{\mathbf{P}}^{n-1}\mid(\xi,\tilde{\xi}^{(i)})_{i\neq m}\in Z_{K_1,K_2,K_3}\}$ is a quadric hypersurface in $\hat{\mathbf{P}}^{n-1}$.

Proof. (1) Let $\{a,b\}=K_1-M$. Then the element $(\xi,\tilde{\xi}^{(i)})_{i\neq a}$ is contained in Z_{K_1,K_2,K_3} if and only if ξ is contained in the linear hull of $L_{K_2}\cap L_{K_3}$ and $\tilde{\xi}^{(b)}$.

(2) Let $m\in M$. For generic $(\tilde{\xi}^{(i)})_{i\in M-m}$, let L be the linear hull of $(\tilde{\xi}^{(i)})_{i\in M-m}$ and $p_1,\dots,p_4$ be independent linear forms wich vanish on L. We write $K_i-M=\{k_{i1},k_{i2}\}$. Let $\hat{\xi}^{(k_{ij})}$ and $\hat{\xi}^{(m)}$ be elements in $\mathbf{C}^n-\{0\}$ and put $\kappa_{ij}^{(m)}=p_m(\hat{\xi}^{(k_{ij})})$ and $\kappa^{(m)}=p_m(\hat{\xi}^{(m)})$. Then $\kappa_{ij}^{(m)}$ and $\kappa^{(m)}$ are linear forms on $\hat{\xi}^{(k_{ij})}$ and $\hat{\xi}^{(m)}$ respecively. Let $\tau^{(i)}$ be an element of $\mathbf{C}^4$ defined by

$$\tau^{(i)}=\det\begin{pmatrix}\kappa^{(1)} & \kappa^{(2)} & \kappa^{(3)} & \kappa^{(4)}\\ \kappa_{i1}^{(1)} & \kappa_{i1}^{(2)} & \kappa_{i1}^{(3)} & \kappa_{i1}^{(4)}\\ \kappa_{i2}^{(1)} & \kappa_{i2}^{(2)} & \kappa_{i2}^{(3)} & \kappa_{i2}^{(4)}\\ \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 & \mathbf{e}_4\end{pmatrix}.$$

where $\mathbf{e}_i$ are the elementary unit vectors. If we put $\xi^{(k_{ij})},\xi^{(m)}$ as the image of $\hat{\xi}^{(k_{ij})},\hat{\xi}^{(m)}$, then $(\xi^{(k_{ij})},\xi^{(m)},\tilde{\xi}^{(i)})_{i\in M-m}$ is contained in Z_{K_1,K_2,K_3} if and only if

(*) the rank of $(\tau^{(1)},\tau^{(2)},\tau^{(3)},\tau^{(4)})$ is less than or equal to 2.

Since

$$(**)\qquad \kappa^{(1)}\det(\tau^{(1)},\tau^{(2)},\tau^{(3)})+\kappa^{(4)}\det(\tau^{(2)},\tau^{(3)},\tau^{(4)})=0,$$

$\det(\tau^{(1)},\tau^{(2)},\tau^{(3)})$ is divisible by $\kappa^{(4)}$. Therefore

$$F(\kappa^{(m)},\kappa_{ij}^{(m)})=\frac{\det(\tau^{(1)},\tau^{(2)},\tau^{(3)})}{\kappa^{(4)}}=-\frac{\det(\tau^{(2)},\tau^{(3)},\tau^{(4)})}{\kappa^{(1)}}.$$

is a polynomial on $\kappa^{(m)},\kappa_{ij}^{(m)}$ of total degree 2 and 1 respectively. Moreover the equality (**) implies the condition (*) is equivalent to $F(\kappa^{(m)},\kappa_{ij}^{(m)})=0$. This proves the assertion (2).

§4.2 Morphism with locally constant fundamental group

In this section, we investigate the local property of the fundamental group for the morphism $X(n,k)\to X(n,k-1)$.

Definition. Let X, Y be smooth varieties, and $f : X \to Y$ be a morphism. f is said to be locally constant in fundamental group if there is a codimension 2 subvariety Σ in Y and smooth (possiblly not connected) divisor D in $Y - \Sigma$ such that

(1) $f\mid_{(Y-\Sigma)-D}: f^{-1}((Y-\Sigma)-D) \to (Y-\Sigma)-D$ is a fiber bundle.

(2) For any point $x \in D$ there exists a contractible open set U in $Y - \Sigma$ containing x such that $\pi_1(f^{-1}(U))$ is isomorphic to $\pi_1(f^{-1}(y))$ for all $y \in U - D$ via the natural inclusion.

Theorem 4.5. *The natural forgetful map $X(n,k) \to X(n,k-1)$ is locally constant in fundamental group.*

Proof. Let $\bar{D} = X(n,k-1) - X(n,k-1)_{SM}$ and Σ be the singular locus of $\bar{D}$. Then $D = \bar{D} - \Sigma$ decomposes into smooth connected components. These components are either

(a) A fiber at the point in this divisor belongs to Manin-Schechtman lattice, or

(b) An open set of Z_{K_1,K_2,K_3} defined in §4.1.

Let $x \in D$. We choose a neighborhood U of x in $X(n,k-1) - \Sigma$ such that $U = \Delta^n, U \cap D = \{0\} \times \Delta^{n-1}$ and $f^{-1}(U)$ is diffeomorphic to the product $X(n,k)_\Delta \times \Delta^{n-1}$, where $X(n,k)_\Delta = f^{-1}(\Delta \times \{0\})$. Note that $X(n,k)_\Delta$ is an open set of $\hat{\mathbf{P}}^{n-1} \times \Delta$. Then we reduce the theorem to the local constantness for

$$f_0 = f\mid_{X(n,k)_\Delta}: X(n,k)_\Delta \to \Delta.$$

Let L be a 2-dimensional linear subspace of $\hat{\mathbf{P}}^{n-1} \times \{0\}$ such that L doesn't intersect codimensiont 3 stratum of $\hat{\mathbf{P}}^{n-1} \times \{s\} - f_0^{-1}(s)$ for all $s \in \Delta$. We choose a tubler neighborhood N of

(a) codimension 3 stratum of $\hat{\mathbf{P}}^{n-1} \times \{0\} - f_0^{-1}(0)$ and $N \cap L = \emptyset$ or,

(b) $(L_{K_1} \cap L_{K_2} \cap L_{K_3}) \cap (\hat{\mathbf{P}}^{n-1} \times \{0\})$.

such that $f_0^{-1}(0) - N$ is a deformation retract of $f_0^{-1}(0)$. Then there exists a sufficiently small ϵ such that

(1) $X(n,k)_\Delta - N \times \Delta_\epsilon$ is a deformation retract of $X(n,k)_\Delta$ and

(2) $f_0^{-1}(\Delta_\epsilon) - N \times \Delta_\epsilon \to \Delta_\epsilon$ is a fiber bundle, where $\Delta_\epsilon = \{z \in \Delta \mid\mid z \mid< \epsilon\}$.

Case (a) By Zariski's theorem, we have the isomorphisms:

$$\pi_1(L \cap f^{-1}(0)) \simeq \pi_1(f^{-1}(0)) \simeq \pi_1(f^{-1}(0) - N) \simeq \pi_1(f_0^{-1}(\Delta_\epsilon) - N \times \Delta_\epsilon)$$

and $\pi_1(L \cap f_0^{-1}(\eta)) \simeq \pi_1(f_0^{-1}(\eta))$ for $\eta \in \Delta - \Delta_\epsilon$. By the exact sequence

$$\begin{array}{ccccc}
\pi_1(L \cap f_0^{-1}(\eta)) & \longrightarrow & \pi_1(L \cap f_0^{-1}(\Delta - \Delta_\epsilon)) & \longrightarrow & \pi_1(\Delta - \Delta_\epsilon) \\
\downarrow \simeq & & \downarrow & & \downarrow \simeq \\
\pi_1(f_0^{-1}(\eta)) & \longrightarrow & \pi_1(f_0^{-1}(\Delta - \Delta_\epsilon)) & \longrightarrow & \pi_1(\Delta - \Delta_\epsilon),
\end{array}$$

we have $\pi_1(L \cap f_0^{-1}(\Delta - \Delta_\epsilon)) \simeq \pi_1(f_0^{-1}(\Delta - \Delta_\epsilon))$. Therefore we can compute the fundamental group of $\pi_1(X(n,k)_\Delta) \simeq \pi_1(X(n,k)_\Delta - (N \times \Delta_\epsilon))$ using van Kampen's theorem:

$$\pi_1(X(n,k)_\Delta) \simeq \pi_1(L \cap f_0^{-1}(\Delta - \Delta_\epsilon)) *_{\pi_1(L \cap f^{-1}(S^1_\epsilon) - N \times S^1_\epsilon)} \pi_1(L \cap f^{-1}(0)),$$

where $S^1_\epsilon = \{z \in \Delta \mid\mid z \mid= \epsilon\}$.

(Figure 5)

Since $\pi_1(L \cap f_0^{-1}(\Delta - \Delta_\epsilon)) \simeq \pi_1(L \cap f_0^{-1}(S^1_\epsilon)) \simeq \pi_1(L \cap f^{-1}(0)) \times \mathbf{Z}$, we have $\pi_1(X(n,k)_\Delta) \simeq \pi_1(L \cap f^{-1}(0)) \simeq \pi_1(L \cap f_0^{-1}(s))$. Therefore it is locally constant in fundamental group.

Case (b). Let $X(n,k)_{\Delta_\epsilon} = f_0^{-1}(\Delta_\epsilon)$, then the inclusion $X(n,k)_{\Delta_\epsilon} \to X(n,k)_\Delta$ is a deformation retract. Moreover if we put $\delta = \epsilon/2$, the inclusion $X(n,k)_{\Delta_\epsilon} - (N \times \Delta_\delta) \to X(n,k)_{\Delta_\epsilon}$ is also a deformation retract. Let $\tilde{N} = N \times (\Delta_\epsilon - \Delta_\delta) \cap f_0^{-1}(\Delta_\epsilon - \Delta_\delta)$ and $B = (\partial N \times (\Delta_\epsilon - \Delta_\delta)) \cap f_0^{-1}(\Delta_\epsilon - \Delta_\delta)$. Since $X(n,k)_{\Delta_\epsilon} - (N \times \Delta_\delta) = \{X(n,k)_{\Delta_\epsilon} - N \times \Delta_\epsilon\} \cup \tilde{N}$, we have

$$(***) \qquad \pi_1(X(n,k)_{\Delta_\epsilon}) \simeq \pi_1(f^{-1}(0)) *_{\pi_1(B)} \pi_1(\tilde{N})$$

by van Kampen's theorem.

(Figure 6)

By Zariski's theorem, we have

$$\begin{aligned} \pi_1(f^{-1}(0)) &\simeq \pi_1(f^{-1}(0) \cap L), \\ \pi_1(\tilde{N}) &\simeq \pi_1(\tilde{N} \cap (L \times (\Delta_\epsilon - \Delta_\delta))) \text{ and} \\ \pi_1(B) &\simeq \pi_1(B \cap (L \times (\Delta_\epsilon - \Delta_\delta))). \end{aligned}$$

We compare the group (***) with the group

$$\pi_1(f_0^{-1}(\delta)) =$$

$$\pi_1((f_0^{-1}(\delta) - (N \times \delta)) \cap (L \times \delta)) *_{\pi_1(B \cap f_0^{-1}(\delta) \cap (L \times \delta))} \pi_1(\tilde{N} \cap f_0^{-1}(\delta) \cap (L \times \delta))$$

via the map induced by

$$\pi_1(f_0^{-1}(\delta) \cap (L - N) \times \delta) \simeq \pi_1(f^{-1}(0)),$$

$$\begin{aligned} &\pi_1(B \cap f_0^{-1}(\delta) \cap (L \times \delta)) = <\alpha, \beta, \gamma \mid \alpha\beta\gamma = \beta\gamma\alpha = \gamma\alpha\beta> \\ &\to \pi_1(B \cap (L \times (\Delta_\epsilon - \Delta_\delta))) = <\alpha, \beta, \gamma, \delta \mid \alpha\beta\gamma = \beta\gamma\alpha = \gamma\alpha\beta, \\ &\qquad [\alpha, \delta] = [\beta, \delta] = [\gamma, \delta] = 1> \end{aligned}$$

and

$$\begin{aligned} &\pi_1(\tilde{N} \cap f_0^{-1}(\delta) \cap (L \times \delta)) = \alpha\mathbf{Z} \oplus \beta\mathbf{Z} \oplus \gamma\mathbf{Z} \\ &\to \pi_1(\tilde{N} \cap (L \times (\Delta_\epsilon - \Delta_\delta))) = \alpha\mathbf{Z} \oplus \beta\mathbf{Z} \oplus \gamma\mathbf{Z} \oplus \delta\mathbf{Z}. \end{aligned}$$

Using the above expression, we have

$$\pi_1(X(n,k)_{\Delta_\epsilon}) \simeq \pi_1(f_0^{-1}(\delta)).$$

§5.1 locally constantness and homotopy exact sequence

In this paragraph, we prove the following proposition.

Proposition 5.1. *Let X, Y be smooth algebraic varieties and $f : X \to Y$ be a smooth morphism with locally constant fundamental group. If the induced homomorphism $\pi_2(X) \to \pi_2(Y)$ of the second fundamental group is surjective, we have the following exact sequence for the fundamental groups for a generic point η in Y.*

$$1 \to \pi_1(f^{-1}(\eta), t) \to \pi_1(X, t) \to \pi_1(Y, \eta) \to 1$$

Proof. Let Σ and D be the codimension 2 subvariety of Y and smooth divisor of $Y - \Sigma$ in the definition of locally constantness. Then $\pi_1(X - f^{-1}(\Sigma)) \simeq \pi_1(X)$ and $\pi_1(Y - \Sigma, \eta) \simeq \pi_1(Y, \eta)$, since the real codimension of Σ is 4. Therefore we may assume $\Sigma = \emptyset$. For the exactness at $\pi_1(X, t)$ and $\pi_1(Y, \eta)$ are proved in [M], we will prove the injectivity for $\pi_1(f^{-1}(\eta), t) \to \pi_1(X, t)$. Let $\gamma : [0, 1] \to f^{-1}(\eta)$ be a loop in $f^{-1}(\eta)$ which is a boundary of a map $\delta : B^2 \to X$, where B^2 is 2-dimensional ball $\{(x, y) \mid x^2 + y^2 \leq 1\}$. By Sard's theorem, we may assume the map δ is transversal to $f^{-1}(D)$, i.e. for $x \in \delta^{-1}(f^{-1}(D))$, $rank(d\delta_x) = 2$ and $Im(d\delta_x) \cap T_{\delta(x)} f^{-1}(D) = 0$. By the assumption of surjectivity for $\pi_2(X) \to \pi_2(Y)$, there exists a map from $\tau : S^2 \times [0, 1] \to Y$ such that $\tau(\xi, 0) = f \circ \delta(\bar{\xi})$, where ξ is the natural image of $\bar{\xi}$ under the map $B^2 \to B^2/\partial B^2 \simeq S^2$ and a map $\tilde{\delta} : S^2 \to X$ such that $\tau(\xi, 1) = f \circ \tilde{\delta}(\xi)$. Again we may asuume that $\tilde{\delta}$ (resp. τ) is transversal to $f^{-1}(D)$ (resp. D) by Sard's theorem. Let X_τ be the fiber product $(S^2 \times [0, 1]) \times_Y X$. Then the natural map $X_\tau \to S^2 \times [0, 1]$ is locally constant in fundamental group by the transversality of τ and there is a induced map $S^2 \to X_\tau$ such that the following diagram commutes.

$$\begin{tikzcd}
S^2 \arrow[r, "(\tilde{\delta}, id \times 1)"] \arrow[d, "\simeq"'] & X_\tau \arrow[d] \\
S^2 \arrow[r, "(id \times 1)"'] & S^2 \times [0, 1]
\end{tikzcd}$$

$$\begin{tikzcd}
\partial B^2 \arrow[r] & B^2 \arrow[r, "(\delta, (id \times 0) \circ pr)"] \arrow[d, "pr"'] & X_\tau \arrow[d] \\
 & S^2 \arrow[r, "id \times 0"'] & S^2 \times [0, 1]
\end{tikzcd}$$

By the second diagram, the image of γ under the map $\pi_1(f^{-1}(\eta)) \to \pi_1(X_\tau)$ is 1.

Claim. *The induced map $\pi_1(f^{-1}(\eta)) \to \pi_1(X_\tau)$ is an isomorphism.*

Proof. To prove the above statement by van Kampen's theorem and triangulation of $S^2 \times [0, 1]$.

(*) For any contructible open set $U_1 \subset U_2$ of $S^2\times[0,1]$ such that $U_1\cap(S^2\times 1)\neq\emptyset$, the induced map

$$\pi_1(U_1, U_1\cap(S^2\times 1)) \to \pi_1(U_2, U_2\cap(S^2\times 1))$$

is an identity.

The claim (*) is also a consequence of van Kampen's theorem and the locally triviality of the fundamental group for $X_\tau \to S^2\times[0,1]$.

§5.2 The exact sequence for the second fundamental group

In this paprgraph, we study the exactness for the second fundamental groups.

Proposition 5.2. *Under the notation in §4.2 , we have the following exact sequences:*

$$\pi_2(f^{-1}(\eta), t) \to \pi_2(X(n,k) - f^{-1}(\Sigma), t) \to \pi_2(X(n,k-1)-\Sigma, \eta)$$

and

$$\pi_2(f^{-1}(\eta), t) \to \pi_2(X(n,k), t) \to \pi_2(X(n,k-1), \eta),$$

where $\eta \in X(n,k-1)$ and $t\in f^{-1}(\eta)$.

Proof. The second exact sequence is a direct consequence of the first because the codimension of $f^{-1}(\Sigma)$and Σ in $X(n,k)$ and $X(n,k-1)$ is 2. (The real codimension is 4.) First we fix an open covering $\{U_i\}$ of $X(n,k-1)-\Sigma$ and choose a local section s_i on U_i. Choose a tubler neibourhood N of

(a) codimension 3 stratum of $\hat{\mathbf{P}}^{n-1}\times\{0\} - f_0^{-1}(0)$ or
(b) $(L_{K_1}\cap L_{K_2}\cap L_{K_3})\cap(\hat{\mathbf{P}}^{n-1}\times\{0\})$,

such that $N\times U_i$ doesn't meet s_i by changing U_i if necessary. Now we consider the map $\delta: D^2 \to X(n,k)-f^{-1}(\Sigma)$ and $\tau: D^2\times[0,1]\to X(n,k-1)-\Sigma$ such that

(1) $\delta(\partial B^2) = \{t\}$,
(2) $\tau(\partial B^2\times[0,1]) = \{\eta\}$, $\tau(B^2\times 1)=\{\eta\}$, $\tau(a,0) = f\circ\delta(a)$.
(3) The maps δ and τ are trnasversal to the divisors $f^{-1}(D)$ and D respectively.

By the condition (3), $\tau^{-1}(D)$ is a smooth curve such that $\partial(B^2\times[0,1])\cap\tau^{-1}(D)\subset B^2\times 0$ For any element $\bar{\delta}$ of the kernel of the map $\pi_2(X(n,k)-f^{-1}(\Sigma),t)\to\pi_2(X(n,k-1)-\Sigma,\eta)$, we can take such a map δ as representative of $\bar{\delta}$. Now we divide $B^2\times[0,1]$ into 3-simplices Δ_i $(i=1,\dots,N)$ such that

(1) $\partial^{(0)}\Delta_m = (B^2\times 0)\cup(\partial B^2\times[0,1])\cup(\cup_{i=1}^{m-1}\Delta_i))\cap\partial\Delta_m$ is homeomorphic to B^2
(2) $\partial^{(1)}\Delta_m = \partial\Delta_m - \partial^{(0)}\Delta_m$ is also homeomorphic to B^2.
(3) $\#(\tau^{-1}(D)\cap\partial\Delta_m) = 0$ or 2 and $\tau^{-1}(D)\cap\partial^{(0)}\Delta_m\cap\partial^{(1)}\Delta_m=\emptyset$.
(4) If $\tau^{-1}(D)\cap\partial^{(0)}\Delta_m = 2$ or $\tau^{-1}(D)\cap\partial^{(1)}\Delta_m = 2$, then there exists a simple path γ connecting $\tau^{-1}(D)\cap\partial^{(0)}\Delta_m$ or $\tau^{-1}(D)\cap\partial^{(1)}\Delta_m$ in $\partial^{(0)}\Delta_m$ or $\partial^{(1)}\Delta_m$ such that γ and $\Delta_m\cap\tau^{-1}(D)$ are homologous in Δ_m.
(5) If $\#(\tau^{-1}(D)\cap\partial^{(0)}\Delta_m) = \#(\tau^{-1}(D)\cap\partial^{(1)}\Delta_m) = 1$, then there exists a homeomorphism $\Delta_m\simeq B^2\times[0,1]$ such that $\Delta_m\cap\tau^{-1}(D)\simeq\{0\}\times[0,1]$, $\partial^{(0)}\Delta_m = \partial\Delta_m\cap(D^2\times[0,1/2])$ and $\partial^{(1)}\Delta_m = \partial\Delta_m\cap(B^2\times[1/2,1])$.

(6) If $\#(\tau^{-1}(D) \cap \Delta_m) = 0$, then $\tau^{-1}(D) \cap \Delta_m = \emptyset$.
(7) At $\tau^{-1}(D) \cap \partial\Delta_m$, τ is transversal for $\partial\Delta_m$ and $\tau^{-1}(D)$.
(8) $f(\Delta_m)$ is contained in some U_i.

On each Δ_m, we prove the following proposition.

Proposition 5.3. *For a given continuous map $\delta_m^{(0)} : \partial^{(0)}\Delta_m \to X(n,k) - f^{-1}(\Sigma)$, there is an extension $\tilde{\tau}_m : \Delta_m \to X(n,k) - f^{-1}(\Sigma)$ which is a lifting of $\tau \mid_{\Delta_m}$.*

$$\begin{array}{ccc} \partial^{(0)}\Delta_m & \xrightarrow{\delta_m^{(0)}} & X(n,k) - \Sigma \\ \downarrow & \nearrow \tilde{\tau}_m & f \downarrow \\ \Delta_m & \xrightarrow[\tau|_{\Delta_m}]{} & X(n,k-1) - \Sigma \end{array}$$

As a corollary, we have the following statement

Corollary 5.4. *(1) The union of $\tilde{\tau}_m$ gives a lifting of τ. (2) $ker(\pi_2(X(n,k) - f^{-1}(\Sigma)) \to \pi_2(X(n,k-1) - \Sigma))$ is generated by $\pi_2(f^{-1}(\eta), t)$.*

Proof of Proposition 5.3. (1) Case $\#(\partial^{(0)}\Delta_m \cap \tau^{-1}(D)) = 2$.

We take a homeomorphism $\Delta_m \simeq B^2 \times [0,1]$ such that (1) $\partial^{(0)}\Delta_m = B^2 \times 0$ (2) $\tau^{-1}(D) \cap (D^2 \times [0,1/2]) = \{p,q\} \times [0,1/2]$. We choose a path $\alpha : p \times [0,1/2] \to f^{-1}(\Delta_m)$ which is a lifting of $\tau \mid_{p\times[0,1/2]}$ such that $\alpha(p,1/2) = s_i(\tau(p,1/2))$. Let $\gamma \times 0$ and $\gamma \times 1/2$ be a line connecting $\{p\times 0, q\times 0\}$ and $\{p\times 1/2, q\times 1/2\}$ respectively, then α extends to a map $\tilde{\alpha}$ from $\gamma \times [0,1/2]$ to $f^{-1}(U_i)$ such that

$$\begin{aligned} \tilde{\alpha} \mid_{\gamma\times 0} &= \delta_m^{(0)} \mid_{\gamma\times 0} \\ \tilde{\alpha} \mid_{\gamma\times 1/2} &= s \circ \tau \mid_{\gamma\times 1/2} \\ \tilde{\alpha} \mid_{p\times[0,1/2]} &= \alpha. \end{aligned}$$

Since $((B^2 - \gamma) \times [0,1/2]) \times_{X(n,k-1)} X(n,k)$ is a fiber bundle over $(B^2 - \gamma) \times [0,1/2]$, $\tilde{\alpha}$ extends to a continuous map $\hat{\alpha}$ from $B^2 \times [0,1/2]$ to $f^{-1}(U_i)$ such that

$$\begin{aligned} \hat{\alpha} \mid_{B^2\times 0} &= \delta_m^{(0)} \\ \hat{\alpha} \mid_{B^2\times 1/2} &= s \circ \tau \mid_{B^2\times 1/2} \\ \hat{\alpha} \mid_{\gamma\times[0,1/2]} &= \hat{\alpha}. \end{aligned}$$

Then $\hat{\alpha} \mid_{B^2\times 1/2}$ is a lifting of $\tau \mid_{B^2\times 1/2}$ to $f^{-1}(U_i) - (N \times U_i)$. Since $f^{-1}(U_i) - (N \times U_i)$ is a fiber bundle over U_i, $\hat{\alpha} \mid_{B^2\times 1/2}$ extends to $B^2 \times [1/2,1]$ as a lifting of $\tau \mid_{B^2\times[1/2,1]}$.

(2) Case $\#(\partial^{(0)}\Delta_m \cap \tau^{-1}(D)) = \#(\partial^{(1)}\Delta_m \cap \tau^{-1}(D)) = 1$.

We take a homeomorphism $\Delta_m \simeq B^2 \times [0,1]$ such that (1) $\partial^{(0)}\Delta_m = B^2 \times 0$, (2) $\tau^{-1}(D) \cap \Delta_m \simeq 0 \times [0,1]$. We choose a path $\alpha : 0 \times [0,1/2] \to f^{-1}(U_i)$ which is a lifting of $\tau \mid_{0\times[0,1/2]}$ such that $\alpha(0,1/2) = s \circ \tau(0,1/2)$ then this α can be extended to a map $\tilde{\alpha} : D^2 \times [0,1/2] \to f^{-1}(U_i)$ such that

$$\begin{aligned} \tilde{\alpha} \mid_{B^2\times 0} &= \delta_m^{(0)} \\ \tilde{\alpha} \mid_{B^2\times 1/2} &= s \circ \tau \mid_{B^2\times 1/2} \\ \tilde{\alpha} \mid_{0\times[0,1/2]} &= \alpha. \end{aligned}$$

Then, $\tilde{\alpha}\mid_{B^2\times 1/2}$ can be regarded as a map to $f^{-1}(U_i)-(N\times U_i)$ so it can be extended to $B^2\times[0,1]$ as a lifting of τ.

(3) Case $\#(\partial^{(0)}\Delta_m\cap\tau^{-1}(D))=0, \#(\partial^{(1)}\Delta_m\cap\tau^{-1}(D))=2$.

We choose a homomorphism $\Delta_m\simeq B^2\times[0,1]$ such that (1) $\partial^{(0)}\Delta_m=B^2\times 0$, (2) $\tau^{-1}(D)\cap(B^2\times[0,1/2])=\emptyset$. We extend $\delta_m^{(0)}$ to the map $\tilde{\alpha}$ from $B^2\times[0,1/2]$ such that $\tilde{\alpha}\mid_{B^2\times 1/2}=s\circ\tau\mid_{B^2\times 1/2}$. Then $\tilde{\alpha}$ can be extended to $B^2\times[0,1]$ in the same way.

§5.3 The exact sequence for the fundamental groups.

We use the same notations as in the last paragraph. In this section, we investigate the exactness of

$$1\to\pi_1(C[1,k-1],p)\to\pi_1(X(n,k),p)\to\pi_1(X(n,k-1),p)\to 1$$

Proposition 5.5. *(1) The homomorphism $\pi_i(C([1,k-1]),p)\to\pi_i(C([1,k-2]),p)$ induced by the natural inclusion $C([1,k-1])\to C([1,k-2])$ is surjective if $k\geq n+2$.*

(2) The natural map $\pi_2(X(n,k),p)\to\pi_2(X(n,k-1),p)$ is surjective if $k\geq n+2$.

(3) The natural map $\oplus_{i=1}^k\pi_2(C([1,k]-i),p)\to\pi_2(X(n,k),p)$ is surjective if $k\geq n+1$.

Proof. (1) The space $C(I)$ is defined as $\{\xi\in\hat{\mathbf{P}}^{n-1}\mid\xi\cup\cup_{i\in I}\xi^{(i)}$ is normal crossing $\}$. So its topological type is independent of the generic choice of $\xi^{(i)}$ for $i\in I$. We take generic $\xi^{(i)}$ for $1\leq i\leq k-1$ and without loss of generality we may assume that $\xi^{(1)}=(1:0:\cdots:0)$, $\xi^{(2)}=(0:1:\cdots:0)\ldots\xi^{(n)}=(0:0:\cdots:1)$ and $\xi^{(k)}=(1:\epsilon:\epsilon^2:\cdots:\epsilon^{n-1})$. Then $\xi^{(k)}\in C([1,k-1])$ for sufficiently small ϵ. Since $C([1,k-1])=\hat{\mathbf{P}}^{n-1}-\cup_{K\in\mathcal{A}(k)}D_K$ and $C([1,k])=\hat{\mathbf{P}}^{n-1}-\cup_{\#K=n-1}D_K$, we consider D_K where $K\ni k, \#K=n-1$. If $K=\{1<\cdots<m<i_{m+1}<\cdots<i_{n-2}<k\}$ with $i_{m+1}>m+1$, then the equation of D_K for $\xi=(\xi_0:\cdots:\xi_{n-1})$ is

$$\det\begin{vmatrix}
1 & \epsilon & \ldots & \epsilon^m & \ldots & \epsilon^n\\
1 & 0 & \ldots & 0 & & \\
0 & 1 & \ldots & 0 & & \\
\vdots & & & \vdots & & \\
0 & 0 & \ldots & 1 & & \\
\xi_0^{(i_{m+1})} & \ldots & & & & \xi_{n-1}^{(i_{m+1})}\\
\vdots & & & & & \vdots\\
\xi_0^{(i_{n-2})} & \ldots & & & & \xi_{n-1}^{(i_{n-2})}\\
\xi_0 & \ldots & & & & \xi_{n-1}
\end{vmatrix}$$

$$=\epsilon^m\det\begin{vmatrix}
0 & \ldots & 0 & 1 & \ldots & \epsilon^{n-m}\\
1 & \ldots & 0 & & & \\
0 & 1 & \ldots & & & \\
\vdots & & \vdots & \vdots & & \\
\ldots & 0 & 1 & 0 & & \\
\xi_0^{(i_{m+1})} & \ldots & & & & \xi_{n-1}^{(i_{m+1})}\\
\vdots & & & & \vdots & \\
\xi_0^{(i_{n-2})} & \ldots & & & & \xi_{n-1}^{(i_{n-2})}\\
\xi_0 & \ldots & & & & \xi_{n-1}
\end{vmatrix}=0$$

Therefore D_K tends to $D_{K'}$ where $K' = \{1, \ldots, m, m+1, i_{m+1}, \ldots, i_{n-2}\}$. If we take an tubler neiborhood N of $\cup_{K' \in \mathcal{A}(k)} D_{K'}$, then D_K is contained in N for a sufficiently small ϵ. Therefore we have the inclusion

$$\hat{\mathbf{P}}^{n-1} - N \subset \hat{\mathbf{P}}^{n-1} - \cup_{\# K = n-1} D_K \subset \hat{\mathbf{P}}^{n-1} - \cup_{K \in \mathcal{A}(k)} D_K$$

for a sufficiently small ϵ and the identity

$$\pi_i(\hat{\mathbf{P}}^{n-1} - N, p) \simeq \pi_i(\hat{\mathbf{P}}^{n-1} - \cup_{K \subset [1,k-1], \# K = n-1} D_K, p),$$

as a consequence, we have the required surjectivity.

We prove the following (2_k) $(k \geq n+2)$ and (3_k) $(k \geq n+1)$ by showing the implications $(3_k) \rightarrow (2_{k+1})$ and $(3_k) \rightarrow (3_{k+1})$.

(2_k) $\qquad \pi_2(X(n,k)) \rightarrow \pi_2(X(n,k-1))$ is surjective

(3_k) $\qquad \oplus_{i=1}^{k} \pi_2(C([1,k] - i)) \rightarrow \pi_2(X(n,k))$ is surjective

By the transitivity and faithfullness of the action of $PGL(n, \mathbf{C})$ on $X(n, n+1)$, we have an isomorphism $PGL(n, \mathbf{C}) \simeq X(n, n+1)$. Under this isomorphism, $C([1,n])$ corresponds to the image of torus $\mathbf{G}_m^n/$ diagonal . Therefore this map coincides with the map

$$\pi_1(\mathbf{G}_m^n / \text{ diagonal }) \simeq \mathbf{Z}^{n-1} \rightarrow \pi_1(PGL(n, \mathbf{C})) \simeq \mathbf{Z}/n\mathbf{Z}$$

and it is surjective. Therefore (3_{n+1}) is true.

Consider the following commutative diagram;

$$\begin{array}{ccccc}
\pi_2(C[1,k]) & \longrightarrow & \oplus_{i=1}^{k+1} \pi_2(C([1,k+1] - i)) & \xrightarrow{\phi} & \oplus_{i=1}^{k} \pi_2(C([1,k] - i)) \\
\simeq \downarrow & & \psi' \downarrow & & \psi \downarrow \\
\pi_2(C[1,k]) & \xrightarrow{\tau} & \pi_2(X(n,k+1)) & \xrightarrow{\phi'} & \pi_2(X(n,k))
\end{array}$$

We proved the exactness of the second row in Proposition 5.2.

$(3_k) \rightarrow (2_{k+1})$

The map ϕ is surjective by (1) and ψ is surjective by (3_k). This implies the surjectivity of ϕ' which means (2_{k+1}).

$(3_k) \rightarrow (3_{k+1})$

For any $x \in \pi_2(X(n,k+1))$, there exists $y \in \oplus_{i=1}^{k+1} \pi_2(C([1,k+1] - i))$ such that $\phi'(x) = \psi\phi(y) = \phi'\psi'(y)$ by (3_k). Therefore $x - \psi(y) \in Ker\phi'$, so there exists $z \in \pi_2(C([1,k]))$ such that $x = \psi'(y) + \tau(z)$ by the exactness of Proposition 5.2. Therefore ψ' is surjective and (3_{k+1}) is proved.

Corollary 5.6. *The morphism $X(n,k) \rightarrow X(n,k-1)$ induces the exact sequence.*

$$1 \rightarrow \pi_1(C([1,k-1]), p) \rightarrow \pi_1(X(n,k), p) \rightarrow \pi_1(X(n,k-1), p) \rightarrow 1.$$

Proof. Since $\pi_2(X(n,k+1)) \rightarrow \pi_2(X(n,k))$ is surjective by Proposition 5.5, the natural map $\pi_1(C([1,k])) \rightarrow \pi_1(X(n,k+1))$ is injective by Proposition 5.1.

Main Theorem 5.7. *$\pi_1(X(n,k),p)$ is isomorphic to $G(n,k)$ defined in §3.2 for $k \geq n+1$.*

Proof. Since the corresponding path $\gamma_{i,K}$ satisfies the relation (2.1), (2.2), (2.3), (3.1), (3.2) and (3.3), we have a natural homomorphism form $G(n,k)$ to $\pi_1(X(n,k),p)$. Let $H(n,k)$ be a subgroup of $G(n,k)$ generated by $\gamma_{k,K}$ ($K \in \mathcal{A}(k)$), then it is a normal subgroup of $G(n,k)$. Since the image of $H(n,k)$ is contained in $\pi_1(C[1,k-1])$, we have the follwing diagram with exact rows.

$$\begin{array}{ccccccccc} 1 \to & H(n,k) & \to & G(n,k) & \to & G(n,k-1) & \to 1 \\ & \iota\downarrow & & (1)\downarrow & & (2)\downarrow & \\ 1 \to & \pi_1(C([1,k-1]),p) & \to & \pi_1(X(n,k),p) & \to & \pi_1(X(n,k-1),p) & \to 1 \end{array}$$

We prove the theorem by induction. For $k = n+1$, since $\pi_1(X(n,n+1)) \simeq \mathbf{Z}/n\mathbf{Z}$ and the map $\pi_1(C([1,n])) \to \pi_1(X(n,n+1))$ is surjective, it is enough to prove that $G(n,n+1) \simeq \mathbf{Z}/n\mathbf{Z}$. In the case of $k = n+1$, by (2.1),(2.2), $\gamma_{i,K}(K \in \mathcal{A}(i))$ are commutative to each other and by (3.1), (3.2), (3.3) $\gamma_{i,L\cup j} = \gamma_{j,L\cup i}$ and all $\gamma_{i,K}$ ($i \in [1,n+1]$, $K \in \mathcal{A}(i)$) are commutative to each other. Therefore we can write γ_M for $\gamma_{i,K}$ if $M = K \cup i$. By the relation (2.3), $G(n,n+1)$ is isomorphic to the quotient of $\oplus_{M \subset [1,n+1], \#M=n} \mathbf{Z}\gamma_M$ by the subgroup generated by $\sum_{i \in N} \gamma_N$ ($i = 1, \dots, n+1$). It is isomorphic to $\mathbf{Z}/n\mathbf{Z}$.

By the induction hypothesis, (2) is an isomorphism. If we write the group generated by $\{\gamma_{k,K} \mid K \in \mathcal{A}(k)\}$ and the relation (2.1), (2.2) and (2.3) by $\tilde{H}(n,k)$, then the the natural composite homomorphism

$$\iota \circ \alpha : \tilde{H}(n,k) \xrightarrow{\alpha} H(n,k) \xrightarrow{\iota} \pi_1(C([1,k-1]),p)$$

is an isomophism by Theorem 2.7. By the definition of the map α it is surjective, therefore α is an isomorphism and ι is also an isomorphism. By the above exact sequence, (1) is an isomorphism.

§5.4 Grassman duality

In this section, we give some remarks on Grassman duality and relationship between $\pi_1(C(I))$ and the Manin-Schechtman higher braid group ([MS]). Let V be an n-dimensional vector space and V^* be the vector space of linear forms on V. $\tilde{X}(n,k)$ be a subset of $(V^* - \{0\})^k$ consisting of $(l_1, \dots, l_k)$ such that the configuration $\cup_{i=1}^k \{l_i = 0\}$ belongs to $X(n,k)$. Then $X(n,k)$ is isomorphic to the quotient of $\tilde{X}(n,k)$ by $\mathbf{G}_m^k$ under the action

$$(l_1, \dots, l_k) \mapsto (\lambda_1 l_1, \dots, \lambda_k l_k) \text{ for } (\lambda_1, \dots, \lambda_k) \in \mathbf{G}_m^k.$$

On $(V^* - \{0\})^k$, $GL(n,\mathbf{C})$ acts diagonally and it induces an action on $\tilde{X}(n,k)$. By the condition for $X(n,k)$, $l_1, \dots l_k$ generates V^* if $(l_1, \dots, l_k) \in \tilde{X}(n,k)$. In other words, if we put $W^* = \oplus_{i=1}^k \mathbf{C}e_i$, the linear map $\Phi(l_1, \dots, l_k)$ defined by $W^* = \oplus_{i=1}^k \mathbf{C}e_i \to V^*; e_i \mapsto l_i$ is surjective. Therefore the kernel of the linear map $\Phi(l_1, \dots, l_k)$ is a $(k-n)$ dimensional linear space in W^*. Therefore we have a map $\tilde{X}(n,k) \ni (l_1, \dots, l_k) \mapsto Ker(\Phi(l_1, \dots, l_k)) \in Gr(k-n, W^*)$ from $\tilde{X}(n,k)$ to the Grassman variety of $(n-k)$ dimen

Lemma 5.8. *The quotient $\tilde{X}(n,k)/GL(n,\mathbf{C})$ is isomorphic to an open set $Gr(k-n,W^*)^0$ consisting of points whose Plücker coordinates are non-zero.*

Since the action of $\mathbf{G}_m^k$ and $GL(n,\mathbf{C})$ on $\tilde{X}(n,k)$ is commutative, we have

$$\mathbf{G}_m^k\backslash Gr(k-n,W^*)^0 \simeq \mathbf{G}_m^k\backslash\tilde{X}(n,k)/GL(n,\mathbf{C}) \simeq X(n,k)/PGL(n,\mathbf{C}) \tag{4.1}$$

Here $PGL(n,\mathbf{C})$ acts on $(\hat{\mathbf{P}}^{n-1})^k$ diagonally and it induces an action on $X(n,k)$. Note that if $k \geq n+1$, then this action is free. We apply Grassman duality to the isomorphism (4.1). Attaching $(k-n)$ dimensional subspace U^* in W^* to $U^{*\perp} = \{\phi \in W = Hom(W,\mathbf{C}) \mid \phi\,|_{U^*}=0\}$, we get an isomorphism between $Gr(k-n,W^*)$ and $Gr(k,W)$. By this isomorphism, the open set $Gr(k-n,W^*)^0$ corresponds to the open set $Gr(k,W)^0$ and the action of $\mathbf{G}_m^k$ is equivariant. Therefore we have an isomorphism

$$Gr(k-n,W^*)^0/\mathbf{G}_m^k \simeq Gr(k,W)^0/\mathbf{G}_m^k$$

and composing the isomorphism (4.1), we have

$$X(n,k)/PGL(n,\mathbf{C}) \simeq X(k-n,k)/PGL(n-k,\mathbf{C}). \tag{4.2}$$

Let us consider the forgetful map and the Grassman duality. Since the forgetful map $X(k-n,k) \ni (\xi_1,\dots,\xi_k) \mapsto (\xi_1,\dots,\xi_{k-1}) \in X(k-n,k-1)$ commutes with the action of $PGL(k-n)$, this induces a map $\bar{X}(k-n,k) \to \bar{X}(k-n,k-1)$ where $\bar{X}(n,k)$ denotes the quotient $X(n,k)/PGL(n,\mathbf{C})$. Using duality isomorphism (4.2), we get a map

$$\bar{X}(n,k) \to \bar{X}(n-1,k-1). \tag{4.3}$$

This map (4.3) is described as follows. For an element $(\bar{\eta}_1,\dots,\bar{\eta}_k) \in \bar{X}(n,k)$, choose a lifting $(\eta_1,\dots,\eta_k) \in X(n,k)$. By the normal crossing condition, the configuration in η_k defiined by $\cup_{i=1}^{k-1}(\eta_i \cap \eta_k) \subset \eta_k$ is also normal normal crossing. If we fix an isomorphism $\phi : \eta_k \simeq \mathbf{P}^{n-2}$, we get a configuration of $(k-1)$ hyperplanes in $\mathbf{P}^{n-2}$ and it defines a point in $\bar{X}(n-1,k-1)$. It is easy to see that this point is independent of the choice of the lifting $(\eta_1,\dots,\eta_k)$ and the isomorphism ϕ we chose.

To describe the Manin-Schechtman higher braid group, we define a variety $MS(n,k,\xi)$ for an element $\xi = (\xi_1,\dots\xi_{k-1}) \in X(n-1,k-1)$ with $\xi_i \subset \mathbf{P}^{n-2}$. The variety $MS(n,k,\xi)$ is defined by the $(k-1)$-tuple $(\hat{\xi}_1,\dots,\hat{\xi}_{k-1})$ of affine hyperplanes in $\mathbf{C}^{n-1} = \{(x_1,\dots,x_{n-1})\}$ such that

(1) if we write the hyperplane as $\hat{\xi} = \{\sum_{j=1}^{n-1} a_{i,j}x_j + a_{i,0} = 0\}$ then, the hyperplane $\{\sum_{j=1}^{n-1} a_{i,j}x_j = 0\}$ in $\mathbf{P}^{n-2}$ is equal to the given ξ_i.

(2) the union $\cup_{i=1}^{k-1}\hat{\xi}_i$ is a normal crossing divisor on $\mathbf{C}^{n-1}$.

The Manin-Schechtman higher braid group is defined as the fundamental group of $MS(n,k,\xi)$. Since the group $\mathbf{C}^* \ltimes \mathbf{C}^{n-1}$ acts on $\mathbf{C}^{n-1}$ by $(x_1\dots,x_{n-1}) \mapsto (\lambda x_1+b_1,\dots,\lambda x_{n-1}+b_{n-1})$ for $(\lambda,b_1,\dots,b_{n-1}) \in \mathbf{C}^* \ltimes \mathbf{C}^{n-1}$, the group $\mathbf{C}^* \ltimes \mathbf{C}^{n-1}$ acts on the variety $MS(n,k,\xi)$.

Proposition 5.9. *(1) Let $\bar{\xi}$ be the image of ξ in $\bar{X}(n,k)$. Then the quotient $MS(n,k,\xi)/\mathbf{C}^* \ltimes \mathbf{C}^{n-1}$ is isomorphic to the fiber of (4.3) at $\bar{\xi}$. As a consequence, it is also isomorphic to the fiber of $\bar{p}: \bar{X}(k-n,k) \to \bar{X}(k-n,k-1)$ at $\bar{\eta}$ where $\bar{\eta}$ is the image of $\bar{\xi}$ under the isomorphism (4.2).*

(2) Let η be an element of $X(k-n,k-1)$ and $\bar{\eta}$ be the image of η in $\bar{X}(k-n,k-1)$ under the natural projection. Then the fiber of the map $p: X(k-n,k) \to X(k-n,k-1)$ at η is isomorphic to the fiber of the map $\bar{p}: \bar{X}(k-n,k) \to \bar{X}(k-n,k-1)$ at $\bar{\eta}$ under the natural projection.

Proof. (1) Let $\tilde{\xi}_0$ be the hyperplane of $\mathbf{P}^{n-1}$ defined by $\{x_0 = 0\}$. The hyperplane $\xi_1, \dots, \xi_{n-1}$ can be considered as hyperplanes in $\tilde{\xi}_0$. The subgroup of $PGL(n, \mathbf{C})$ consists of element which fixes all the points in $\tilde{\xi}_0$ is denoted by H. Then the fiber of the map $\bar{X}(n,k) \to \bar{X}(n-1,k-1)$ at $(\xi_1 \dots, \xi_k)$ is isomorphic to the quotient of the set of hyperplanes $(\tilde{\xi}_1, \dots, \tilde{\xi}_k)$ such that (1) $\tilde{\xi}_i \cap \tilde{\xi}_0 = \xi_i$ and (2) $\cup_{i=1}^{k-1} \tilde{\xi}_i$ is a normal crossing divisor of $\mathbf{P}^{n-1}$ by the action of the group H. Using inhomogeneous coordinate, it is isomorp hic to $MS(n,k,\xi)/(\mathbf{C}^* \ltimes \mathbf{C}^{n-1})$.

(2) First we show that any element $\bar{\tau} \in \bar{p}^{-1}(\bar{\eta})$ is in the image of p^{-1}. If we take a representative $\tau = (\tau_1, \dots, \tau_k)$ in $X(n,k)$, then there exists $g \in PGL(n, \mathbf{C})$ such that $g(\tau_1, \dots, \tau_{k-1}) = (\eta_1, \dots, \eta_{k-1})$. Therefore $g\tau$ is contained in $p^{-1}(\tau)$ which represents $\bar{\tau}$. If $\tau^{(1)} = (\tau_1^{(1)}, \dots, \tau_k^{(1)})$ and $\tau^{(1)} = (\tau_1^{(1)}, \dots, \tau_k^{(1)})$ be elements in $p^{-1}(\eta)$ which represent the same $\bar{\tau}$, then there exists $g \in PGL(n, \mathbf{C})$ such that $g\tau^{(1)} = \tau^{(2)}$ and $\tau_i^{(1)} = \tau_i^{(2)} = \eta_i$ for $1 \le i \le k-1$. Therefore $g = 1$ and we get the theorem.

The fiber of p at η is denoted by $C([1,k-1],\eta)$ as in §2. By using the homotopy exact sequence for the fibration $MS(n,k,\xi) \to C([1,k-1],\eta)$, we have the following exact sequence

$$\mathbf{Z} = \pi_1(\mathbf{C}^* \ltimes \mathbf{C}^{n-1}) \to \pi_1(MS(n,k,\xi)) \to \pi_1(C([1,k-1],\eta)) \to 1.$$

The image of the generator in $\pi_1(\mathbf{C}^* \ltimes \mathbf{C}^{n-1})$ under the above map is denoted by δ. Therefore we get the following theorem.

Theorem 5.10. *The group $\pi_1(C([1,k-1],\eta))$ is isomorphic to the quotient of the Manin-Schechtman higher braid group by the cyclic group generated by δ.*

Remark. Let i be an integer such that $1 \le i \le k$. Then the set $\{\gamma_{i,K}\}_{K \in \mathcal{A}(i)}$ is a set of generators of $MS(n,k,\xi)$ and relations are given by (2.2) and (2.3) (see [MS],[L]). The left hand side of (2.3) is equal to δ.

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